On the structure of the (3n+1)/2d(n) iteration problem Part I: Prediction of forward iterations

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Abstract

To investigate the iteration of the Collatz function, we define an operation between periodic integer series that produce arithmetically subsets of them. This operation allows to decompose any periodic integer series along their generalized evenness (the number of times an integer can be divided by 2). For any periodic integer series the same regular (periodic) fractal structure is obtained. Writing how the parameters of this structure are changed through the iteration of the Collatz function, which can be simply drawn, explains the origin of the stochastic appearance of the iterations. It also allows to describe fully these iterations, and to find a general expression for them, even if still in an iterated form for the parameters. This extends the theorem of Lagarias (1985) on the periodicity of numbers of similar history. If we define the history of an integer by the successive evenness through the Collatz function iteration, and compute the number corresponding to a given history, we find that only few histories do not lead to an infinite number.

1 - Introduction :

The Collatz problem (Collatz 1986, Lagarias 2011) is the iterations of the operations defined on the positive integers n:
- if n is even, it is divided by 2
- if n is odd, it is multiplied by 3 and added 1

The real difficulty of this problem is that we do not know a priori how many times we can divide an even number before reaching an odd number.

We thus define the generalized "***evenness***" $e(n)$ of an integer n as the number of times it can be divided by 2 before becoming odd:
$e(n) = q \Leftrightarrow n = 2^q * p$ with p odd.

The iteration can then be rewritten into:
- if n is even, it is divided by $2^{e(n)}$, to give the odd number $p = n/2^{e(n)}$,
- transform the odd number p to $3p+1$, noted by the function $T(p) = 3p+1$

The next step will be to look at how many times $3p+1$ can be divided by 2, in other words we can define the "***iterate evenness***", the function $d(n)$, by :

$d(n) = e(3n+1)$

With $d(n)$ the iteration problem can be resumed to the iteration of the function *C*, defined on odd numbers, by:
$C(n) = (3n+1)/2^{d(n)}$

This is a more compact form than usually used (Lagarias 2011, Lagarias 2012), using the generalized evenness to reduce all the successive division by 2 in one step. The main problem is now to understand the structure of this function $d(n)$. Starting from an even number we know that $d(n) = 0$, but starting from an odd number we just know that $d(n) \geq 1$.

If we write $\overset{m}{C}(n) = \{n, C(n), C^2(n), C^3(n), ..., C^m(n)\}$ the *m* successive iterate of *C* on *n*, the first particular case is $\overset{\infty}{C}(1) = \{\underline{1}...\}$, meaning an infinite repetition of 1, as $C(1) = 1$. A not so special example is $\overset{\infty}{C}(7) = \{7, 11, 17, 13, 5, \underline{1}...\}$, showing that it eventually goes to 1, and then repeat indefinitely. The Collatz conjecture is that every number eventually goes to 1 (Lagarias 1985).

Together with the successive iteration values, let us define the "***history***" of a number thought the iterations of *C*, as the successive *iterate evenness* d. For instance the number 1 has the history $\overset{\infty}{H}(1) = \{\underline{2}...\}$, as $T(1) = 2^2 * 1$. Our other example is $\overset{\infty}{H}(7) = \{1, 1, 2, 3, 4, \underline{2}...\}$. The history is similar to the 'parity vector' defined and used by Terras (1976) and Lagarias (1985), except it si more compact as grouping the successive sequence of even numbers in one, and omitting the odd steps. This is a first interest of the generalized evenness.

To describe a trajectory the best is to write simultaneously the values and history:
$\overset{\infty}{H}/C(7) = \left\{7, \frac{1}{11}, \frac{1}{17}, \frac{2}{13}, \frac{3}{5}, \frac{4}{1}, \frac{2}{\underline{1}}, ...\right\}$, with the understanding that we are not writing fractions, but, after the initial vaule, just $d(n)$ over $C(n)$.
We can represent graphically the histories of the first 256 odd numbers:

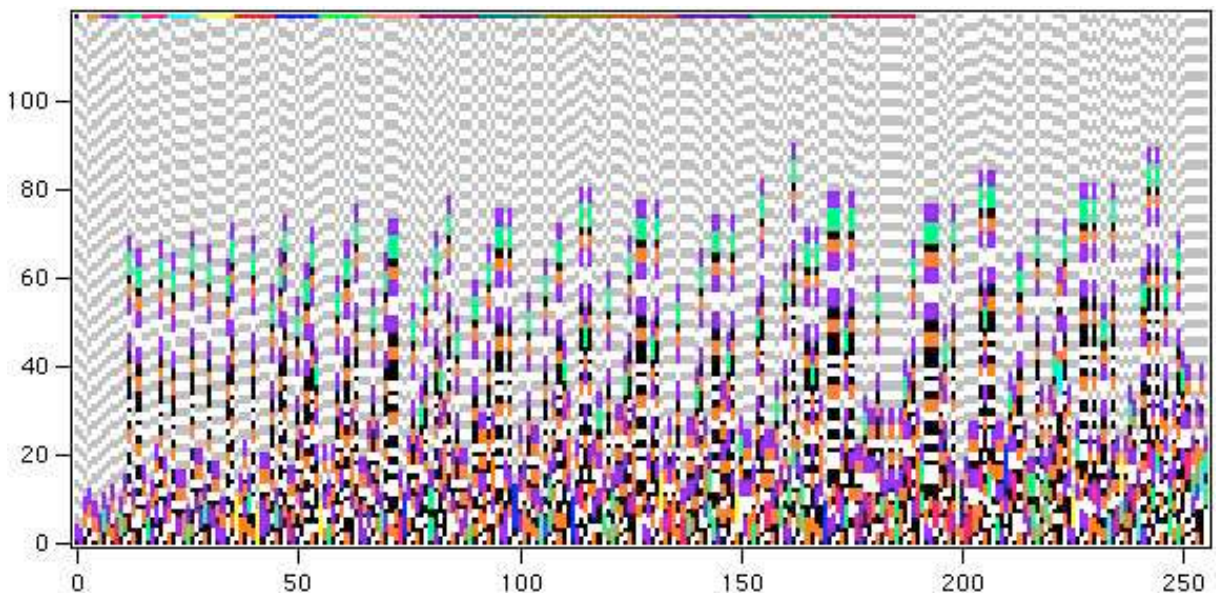

Fig. 1-Numbers histories. Each pixel on the horizontal axis corresponds to the starting odd number (its odd index), on the vertical axis to the cumulated d. Each value of d is drawn with d pixels of a particular colour (d=1black, d=2grey,

d=3orange, d=4violet, ...) as indicated on the top line. When the number has reached the value 1, the rectangle of length 2 that repeats indefinitely is drawn alternatively grey and white to reveal the period and neighbour phase.

Among other things (like the same ending of close numbers, and the general repetition of the same endings) one can notice the special case of number 27, with a sudden jump in the length of total history before reaching 1:

$$\overset{\infty}{H}/C(27)=\Big\{27,\frac{1}{41},\frac{2}{31},\frac{1}{47},\frac{1}{71},\frac{1}{107},\frac{1}{161},\frac{2}{121},\frac{2}{91},\frac{1}{137},\frac{2}{103},\frac{1}{155},\frac{1}{233},\frac{2}{175},\frac{1}{263},\frac{1}{395},\frac{1}{593},\frac{2}{445},\frac{3}{167},\frac{1}{251},\frac{1}{377},\frac{2}{283},$$
$$\frac{1}{425},\frac{2}{319},\frac{1}{479},\frac{1}{719},\frac{1}{1079},\frac{1}{1619},\frac{1}{2429},\frac{3}{911},\frac{1}{1367},\frac{1}{2051},\frac{1}{3077},\frac{4}{577},\frac{2}{433},\frac{2}{325},\frac{4}{61},\frac{3}{23},\frac{1}{35},\frac{1}{53},\frac{5}{5},\frac{4}{1},\frac{2}{\underline{1}},\dots\Big\}$$

Up to now Collatz's conjecture is not proven, even with the numerous work done from very diverse origin (Queneau 1972), analyzed and compiled by Lagarias (Lagarias 2011, Lagarias 2012). The purpose of this work is to show explicitly the underlying structure of this iteration problem.

To this we define an operation among periodic set of integers that allow to take subset of them. This operation allows to write easily the evenness of each term of the series, and decompose it in periodic series of given evenness. We present and develop first this odd-even decomposition, which reveal always the same periodic fractal structure, up to a finite series of parameters. This structure and decomposition present some interesting general properties, and open further questions and generalizations.

With this tool the first 2 iterations are extensively examined. The presentation could be shorter but it was made so to be sure that people not used to this set operation could follow step by step. The next 2 iterations (up to 4) are presented more briefly, together with a simple graphical way to summarize them. Then a general formulation of these iterations is obtained. It states the periodic series of numbers that have the same beginning of history. It performs the idea of Terras (1976) that knowing the beginning of history is enough information to define a set of numbers sharing this beginning of history. In this case it gives an explicit (even though iterative) way to construct theses series. The first examples on 2 and 3 iterations are explicitly given. The general formula recover the result on the periodicity of such sets obtained by Lagarias (1985), and extends it as giving also the first number of these series.

The condition under which such numbers increase with the increase in length of history is then examined, and this lead to a final discussion on the limitation of possible histories. In the next part, we will study more extensively the common structure of the possible histories.

2 - Other Notations and Definitions

We are working on positive integers N, so when a number i is written without specifications it is implicitly for $i \in N$. For series of numbers depending on others, such as a number *a* depending on *i* and *j*, written conventionally as $a(i, j)$, we will essentially for the sake of space write it also $a_{i,j}$.

We will look at ***periodically spaced series*** defined and noted by
$S(g, p) = \{g + k * p, k \in N\}$,
where g is called the "***generator***", p the "***period***", and k the "***index***".

A "***full***" series is such that $0 \le g < p$ (the generator is strictly smaller than the period).

This is very close to the modularity, i.e. $S(g, p) = \{n, n = g[p]\}$, with the condition that we allow "*not full*" series, and that we keep the information about the *index k*, which will prove useful.

On the ensemble of such series (not necessarily *full*) we can directly rewrite the addition and multiplication operations:
$a + S(g, p) = \{a + g + k * p, k \in N\} = S(a + g, p)$
$a * S(g, p) = \{a * g + a * k * p, k \in N\} = S(a * g, a * p)$

In rough words we can see that the generator is "related to the addition" while the period is "related to the multiplication". This is also an important difference with common modularity, as the period is also changed in the operation.

This allows to apply any arithmetic function (like T) to the series directly, together with other ensemble operations, such as:
$T(S(p,q) \cup S(r,s)) = T(S(p,q)) \cup T(S(r,s)) = S(T(p), 3*q) \cup S(T(r), 3*s)$

To introduce a very useful operation and its notation, let us look at the intersection of two *full* series:
$S(p,q) \cap S(r,s)$, with $g = \gcd(q, s)$,
so that $q = g * q'$, $s = g * s'$ and $l = g * q' * s' = lcm(q, s)$.

If there is a not null intersection, it means there is a number u common to both original series:
$u = p + r' * q = r + p' * s$.

From this we can built a periodically spaced series of solution with period $l = g * q' * s'$, as $\forall k \in N$, $u + k * g * q' * s' = p + (r' + k * s') * q = r + (p' + k * q') * s$ is a member of both original series.

This series is a *full* series as, if $u \ge g * q' * s'$, then
$0 \le u - g * q' * s' = p + (r' - s') * q = r + (p' - q') * s$ is also a member of the intersection, so we can repeat the subtraction until we find the good generator such that
$0 \le u' = u - n * g * q' * s' < g * q' * s'$. A direct way to find it is to use the formula:

$u' = u - (g*q'*s')*E\left[u/(g*q'*s')\right]$, where E stands for the integer part. As $u = p + r'*q = r + p'*s$, $u < g*q'*s'$ also implies that $r' < s'$ and $p' < q'$.

This series, if it exist, is the only solution, as can be shown by the absurd. If there was another intersection v not part of this series, we could write $v = p + a*q = r + b*s$, with $a \neq r'$ and $b \neq q'$. We can also assume that $v < g*q'*s'$ by constructing as above a smaller value until this is true. Then we can see that $v - u = (p + a*q) - (p + r'*q) = (r + b*s) - (r + p'*s)$, so that $l' = (a - r')*q = (b - p')*s$ is a new common multiple of q and s (and positive, by taking $\pm l'$). Now we have $r' < s'$ and $a < s'$, as seen above, so we have $0 < |a - r'| < s'$ and $0 < |b - p'| < q'$, which means that $0 < |l'| < l$, which is contradictory with the assumption that l is the lowest common multiple of q and s .

We can thus write :
$S(p,q) \cap S(r,s) = S(u, g*q'*s')$
or $S(p,q) \cap S(r,s) = S(p + r'*q, s'*q) = \left\{p + (r'+k'*s')*q, k' \in N\right\}$
or $S(p,q) \cap S(r,s) = S(r + p'*s, q'*s) = \left\{r + (p'+k'*q')*s, k' \in N\right\}$
This intersection can thus be seen as a periodic subset of each *index* of the two original series, each index subset being defined by the indexes being part of another *full* periodic series:
$S(p,q) \cap S(r,s) = \left\{p + k*q, k \in S(r',s')\right\}$
$S(p,q) \cap S(r,s) = \left\{r + k*s, k \in S(p',q')\right\}$

We define this operation by the term "***subsetting***", and note it this way:
$\left\{p + k*q, k \in S(r',s')\right\} = S(p,q) // S(r',s')$

The previous intersection can thus be written as *subsetting* of each original series:
$S(p,q) \cap S(r,s) = S(p,q) // S(r',s')$
$S(p,q) \cap S(r,s) = S(r,s) // S(p',q')$
Note that the writing is not symmetric (the operation is in general not commutative), and that the expressions of (r',s') and (p',q'), when they exist, are in general not simple.

Property 0 (neutral element)
The ***neutral element*** of this *subsetting* operation is $N = S(0,1)$ itself:
$S(p,q) // S(0,1) = S(p,q) = S(0,1) // S(p,q)$

Property 1 (particular commutations)
The *subsetting* is in general not a commutative operation. However it becomes so when restricted to full series with extreme generators:
• with maximum generators:
$S(q-1,q) // S(s-1,s) = S(-1 + q*s, q*s) = S(s-1,s) // S(q-1,q)$
• with minimum generators (in which case it corresponds simply to the multiplication):
$S(0,q) // S(0,s) = S(0, q*s) = S(0,s) // S(0,q)$

We can see easily how this *subsetting* combines with arithmetic operations:
$a + \left(S(p,q) // S(r,s)\right) = S(a + p, q) // S(r,s) = \left(a + S(p,q)\right) // S(r,s)$

$a*\left(S(p,q)//S(r,s)\right) = S(a*p,a*q)//S(r,s) = \left(a*S(p,q)\right)//S(r,s) = S(a*p,q)//(a*S(r,s))$
and in particular it gives
$T\left(S(p,q)//S(r,s)\right) = T\left(S(p,q)\right)//S(r,s)$

In general we can see that subsetting just takes a subset of the possible indexes, so that all the arithmetic operation on the original series can be done independently of the subsetting of the indexes, which can still be done identically:

Property 2 (arithmetic neutrality)
Any arithmetic operation A is neutral on the subsetting operation, in other words :
$A\left(S(p,q)//S(r,s)\right) = A\left(S(p,q)\right)//S(r,s)$

Similarly we can see how subsetting can combines with itself:

Property 3 (self-distributivity)
$\left(S(p,q)//S(r,s)\right)//S(u,v) = S(p,q)//S(r,s)//S(u,v) = S(p,q)//\left(S(r,s)//S(u,v)\right)$

Property 4 (fullness preservation)
This *subsetting* preserves the "*fullness*", i.e. the subset of a full series by a full series is also a full series (if $0 \le p < q$ and $0 \le r < s$, then $p \le q-1$ and $r \le s-1$, so $S(p,q)//S(r,s) = S(p+r*q, s*q)$ will have $p+r*q \le q-1+(s-1)*q = s*q-1 < s*q$)

Property 5
Similarly, T and the Collatz function C preserves the fullness:
the multiplication preserves the fullness, so the division by a power of 2 (as long as it remains within the integers), will preserve the fullness. This is not the case for the addition, but T in particular preserves the fullness (if $S(p,q)$ is full $p \le q-1$ so $3*p+1 \le 3*q-2 < 3*q$ so $T(S(p,q)) = S(3*p+1, 3*q)$ is full).

We have to be carefull when using ensemble functions, as the subsetting applies only for a well defined series of index, and has not necessarily a meaning for a reunion of different series such as in $\left(S(p,q) \cup S(r,s)\right)//S(t,u)$.

3 - A first example

A first hint of the usefulness of these writings can be obtained when we look at the *evenness* of all the integers. To show that N is a mixing of all the possible evenness numbers, we can decompose $N = S(0,1)$ in two parts, by doubling the period:
$S(0,1) = S(0,2) \cup S(1,2)$
Now $S(1,2)$ are all the odd numbers: $e(S(1,2)) = 0$,
and we can write $S(0,2) = 2*S(0,1) = 2*N$
So we can repeat the decomposition:
$N = \left(2*S(0,1)\right) \cup S(1,2) = \left[2*\left(2*S(0,1) \cup S(1,2)\right)\right] \cup S(1,2)$, so
$N = 2^2*S(0,1) \ \cup \ 2*S(1,2) \ \cup \ S(1,2)$.
And then, iterated to infinity (and 0 having an infinite evenness, $\{0\} = 2^\infty * S(0,1)$):
$$N = \{0\} \bigcup_{i \in N} 2^i * S(1,2)$$

This writing allows to show directly that N contains all the possible evenness, but we seem to have lost the actual decomposition, i.e. the information of which number has a given evenness (the *indexes*). We keep this information if we write the same decomposition as *subsets* of the original series (using *property 0*):
$S(0,1) = S(0,2) \cup S(1,2) = S(0,1)//S(0,2) \;\cup\; S(0,1)//S(1,2)$
We can then iterate it for the first subset:
$S(0,1) = \left(S(0,1)//S(0,2)\right)//S(0,2) \;\cup\; \left(S(0,1)//S(0,2)\right)//S(1,2) \;\cup\; S(0,1)//S(1,2)$, or
$S(0,1) = S(0,1)//S(0,2)//S(0,2) \;\cup\; S(0,1)//S(0,2)//S(1,2) \;\cup\; S(0,1)//S(1,2)$
Now using that the subsetting is distributive (*property 3*), and that
$S(0,2)//S(0,2) = S(0,2^2) = 2^2 * S(0,1)$, and
$S(0,2)//S(1,2) = S(2,2^2) = 2 * S(1,2)$,
it gives, iterated to infinity:
$$N = \{0\} \bigcup_{i \in N} S(0,1)//2^i * S(1,2)$$
Now we have kept the *indexes* information, as we kept the decomposition as a union of *subset* of the original series. Note that in this case instead of repeating the decomposition keeping the index subsetting, we could have directly transformed the first expression into the subset one using *property 0*: $S(p,q) = S(0,1)//S(p,q)$.

This decomposition with a simple writing expression correspond in fact to a perfectly regular fractal structure, as can be seen by drawing $e(n)$ with the same encoding as above:

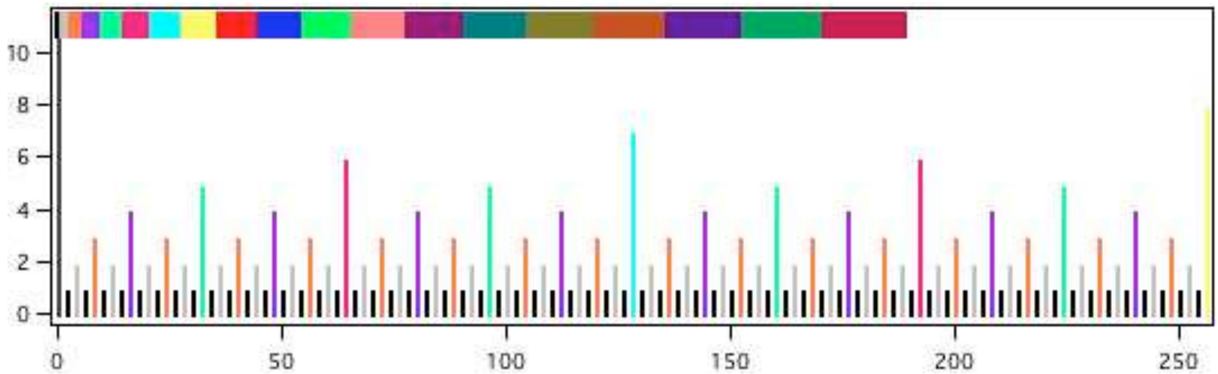

Fig 2 – Evenness of first 256 integers. Drawing as in the previous fig. 1: when evenness is 0, nothing is drawn, for 1, a black (elongated) pixel, for 2, two greys, etc. (code on top left). Zero can be considered as infinite evenness.

4 - Odd/even decomposition

This decomposition of N can seem obvious, but we will just repeat the same process on other series $S(p,q)=T(S(m,n))$, to write explicitly the evenness of each member of this series, so that we can apply the second part of the iteration of C. All depends on the parities of the period q and generator p.

-•- if p is odd ($p=1+2p'$) and q even ($q=2q'$), it is a subset of the odd numbers :
$S(p,q)=S(1,2)//S(p',q')$
So the evenness of all the series is null: $e(S(1+2p',2q'))=0$

-•- if p is even ($p=2p'$) and q even ($q=2q'$), then $S(p,q)=2*S(p',q')$it is a subset of the even numbers, but of possibly various evenness. We have to compare the evenness of p and q.
- If $e(p)<e(q)$ dividing it by $2^{e(p)}$ we are back to the previous case, and all the series have evenness $e(p)$: $e(S(p,q))=e(p)$
- If $e(p)\geq e(q)$ dividing it by $2^{e(q)}$ will give an odd period, which are the next two cases:

-•- if p is odd ($p=1+2p'$) and q odd ($q=1+2q'$), then we can decompose in 2:
$S(p,q)=S(p,q)//S(0,2)\ \cup\ S(p,q)//S(1,2)$ with
$S(p,q)//S(0,2)=S(1+2p',2q)=S(1,2)//S(p',q)$ is a subset of the odd numbers while
$S(p,q)//S(1,2)=S(2+2(p'+q'),2q)=2*S(1+p'+q',q)$ are all even,
and the decomposition has to be continued on this new series,

-•- if p is even ($p=2p'$) and q odd ($q=1+2q'$), then we can decompose in 2:
$S(p,q)=S(p,q)//S(0,2)\ \cup\ S(p,q)//S(1,2)$ with
$S(p,q)//S(0,2)=S(2p',2q)=2*S(p',q)$ are all even while now
$S(p,q)//S(1,2)=S(1+2(p'+q'),2q)=S(1,2)//S(p'+q',q)$ is a subset of the odd numbers,
and the decomposition has to be continued on the new even series.

The interesting case is when the period q is odd. To find the evenness is then based on the ***odd/even decomposition*** of the indexes :
$S(p,q)=S(p,q)//S(0,2)\ \cup\ S(p,q)//S(1,2)$.
We found that it also correspond in the case of odd period to a decomposition of the seris in two parts, one odd and one even.
Using the fact that it is constructed on the subestting operation, we have already a first property:

Property 6
The *odd/even decomposition* $S(p,q)=S(p,q)//S(0,2)\ \cup\ S(p,q)//S(1,2)$ preserves the fullness, i.e. each two series are full, as $S(0,2)$ and $S(1,2)$ are full, and the subsetting preserve the fullness (*property 4*).

Then the result of the decomposition, depending on the parity of the generator p, can be summarized by:

$$S(p,q),\quad q=1+2q' \quad \begin{cases} p=1+2p', & S(p,q)=2*S(1+p'+q',q)\ \cup\ S(1,2)//S(p',q) \\ p=2p', & S(p,q)=2*S(p',q)\ \cup\ S(1,2)//S(p'+q',q) \end{cases}$$

Property 7 (odd-even decomposition iteration)
We see that the result of the odd/even decomposition of a series $S(p,q)$ with an odd period $q = 1 + 2q'$ can always be written as

$S(p,q) = 2 * S(p^+, q) \cup S(1,2) // S(a,q)$

with a the generator of the subset of the odd numbers (of period q), and p^+ the (half) generator of the even series (also of -half- period q), just depending on the parity of p:

$$\begin{cases} p = 1 + 2p': & a = p', \quad p^+ = 1 + p' + q' \\ p = 2p': & a = p' + q', \quad p^+ = p' \end{cases}$$

This formula gives for a given p the corresponding odd generator a and next value p^+, from which we can compute a^+, and so on and so forth. As there is a direct relationship between p and a, we can try to write explicitly the relation from a to a^+ by iterating it a second time (as we have the same odd period q):

$$\begin{cases} p = 1 + 2p': \quad a = p', \quad p^+ = 1 + p' + q' = a + 1 + (q-1)/2 & \begin{cases} p^+ = 1 + 2p^{+\prime}: & a^+ = p^{+\prime} = (p^+ - 1)/2 = a/2 + (q-1)/4 \\ p^+ = 2p^{+\prime}: & a^+ = p^{+\prime} + q' = p^+/2 + (q-1)/2 = (a+1)/2 + 3(q-1)/4 \end{cases} \\ p = 2p': \quad a = p' + q', \quad p^+ = p' = a - (q-1)/2 & \begin{cases} p^+ = 1 + 2p^{+\prime}: & a^+ = p^{+\prime} = (p^+ - 1)/2 = (a-1)/2 - (q-1)/4 \\ p^+ = 2p^{+\prime}: & a^+ = p^{+\prime} + q' = p^+/2 + (q-1)/2 = a/2 + (q-1)/4 \end{cases} \end{cases}$$

In the explicit formulas of the subsetting generators a^+, it is not obvious that we obtain integers, in particular for $(q-1)/4$: we only know that q is odd, so $e(q-1) \geq 1$, but its evenness can be equal to 1. Similarly for $a/2$, as a can be even as well as odd. However, we know that all the above generators are integers, *by construction*. This knowledge is a way to shortcut the discussion on the parities of p and p^+. The other distinction we can notice, is that that if $p = 1 + 2p' < q$, then $a = (p-1)/2 < (q-1)/2$, while if $p = 2p' \geq 0$, then $a = p/2 + (q-1)/2 \geq (q-1)/2$. With that we can write directly $a^+(a)$ separating the different cases:

Property 8 (odd index "a" iterations)
In the successive explicit evenness decomposition, the next generator a^+ of the subset of the odd numbers can be computed from the previous one a (and the odd period q) with:

$$\begin{cases} e(q-1) = 1 & \begin{cases} e(a) = 0: & a^+ = a/2 + (q-1)/4 \\ e(a) \geq 1 & \begin{cases} a < (q-1)/2: & a^+ = (a+1)/2 + 3(q-1)/4 \\ a \geq (q-1)/2: & a^+ = (a-1)/2 - (q-1)/4 \end{cases} \end{cases} \\ e(q-1) \geq 2 & \begin{cases} e(a) = 0 & \begin{cases} a < (q-1)/2: & a^+ = (a+1)/2 + 3(q-1)/4 \\ a \geq (q-1)/2: & a^+ = (a-1)/2 - (q-1)/4 \end{cases} \\ e(a) \geq 1: & a^+ = a/2 + (q-1)/4 \end{cases} \end{cases}$$

We can rewrite the expressions of a^+ :

$$
\begin{cases}
e((q-1)/2)=0 & \begin{cases} e(a)=0: & a^+ = a/2+(q-1)/4 \\ e(a)\geq 1 & \begin{cases} a<(q-1)/2: & a^+ = (a+q)/2+(q-1)/4 \\ a\geq (q-1)/2: & a^+ = (a-q)/2+(q-1)/4 \end{cases} \end{cases} \\
e((q-1)/2)\geq 1 & \begin{cases} e(a)=0 & \begin{cases} a<(q-1)/2: & a^+ = (a+q)/2+(q-1)/4 \\ a\geq (q-1)/2: & a^+ = (a-q)/2+(q-1)/4 \end{cases} \\ e(a)\geq 1: & a^+ = a/2+(q-1)/4 \end{cases}
\end{cases}
$$

so that it is always the same linear function $f_q(a) = a/2+(q-1)/4$, with only a shift in the abscissa by q, $f_q(a\pm q)$, depending on the parities: if a and $(q-1)/2$ have both null or not null parity, $a^+ = f_q(a)$, if not, then if $a<(q-1)/2$, $a^+ = f_q(a+q)$, and if $a>(q-1)/2$, $a^+ = f_q(a-q)$. *Figure 3* shows this straight line of slope ½ wrapped with periodic boundaries distant by q. Even wrapped, it is not a multivalued function because each parity, lower or larger than half the interval, corresponds to a unique piece of line. The different conditions expressed above are just to ensure that each piece of line corresponds to the good parity in order that the result is still an integer. In particular on the line only one point every 2 are selected (to be integer because of the ½ slope), and this period is kept at the periodic boundary. Because q is odd, this give once wrapped the other parity for a. This function is a bijection, each possible origin has a unique image and each possible image in the interval has a unique origin.

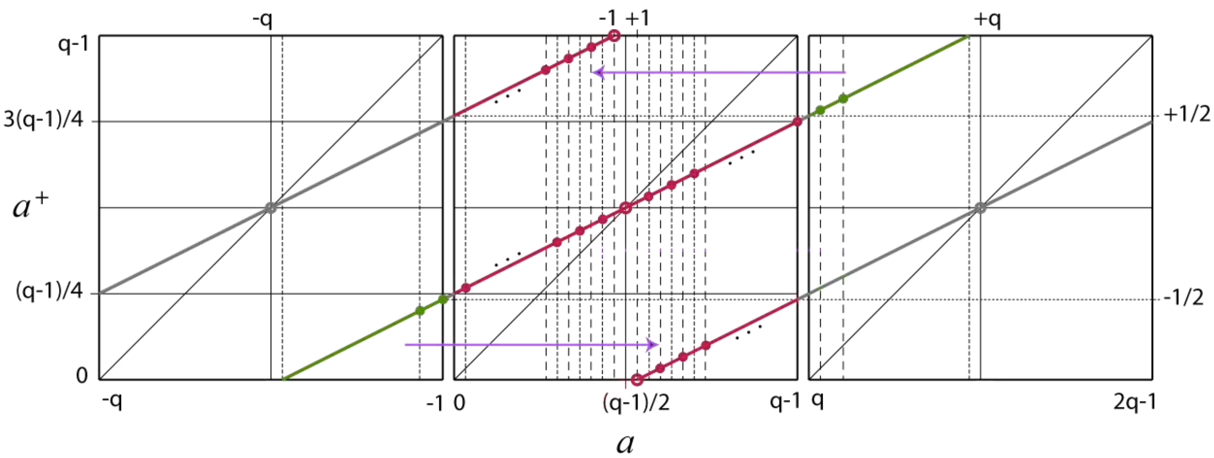

Fig. 3- The iteration function giving the new subsetting odd number generator (a^+) in the odd-even decompositions of a series $S(p,q)$ with q odd, as a function of the previous one (a). The points are regularly placed around the fixed point $(q-1)/2$. When crossing the border, the last point is either on the border of the interval (right, for $(q-1)/2$ even), or this is the next point which is on the border of the next interval (left, for $(q-1)/2$ odd). Of course for a given q, the real figure have a perfect central symmetry.

We see that during this decomposition we obtain again a series of even numbers, which once divided by 2, gives a new series $S(p^+,q)$ *with the same odd period q*. Thus it has to be decomposed exactly in the same way, infinitely:

Property 9 (explicit evenness decomposition)
A series $S(p,q)$ with the period q odd contains all the evenness.
The decomposition will be of the form of subsets of the odd numbers with period q and explicit evenness i:
$$S(p,q)=\bigcup_{i\in N} 2^i * S(1,2)//S(a_i,q)$$
The generator series a_i can be computed with *property 7* or *8*.

As each of this subset is a periodic series part of the original one, we can also see this decomposition as subsets of *the original series*:
$$S(p,q)=\bigcup_{i\in N} S(p,q)//S(x_i,q'_i)$$
Equalling the 2 expressions $S(p,q)=\bigcup_{i\in N} S(p+x_i*q,q*q'_i)=\bigcup_{i\in N} S(2^i+2^{i+1}*a_i,2^{i+1}*q)$ leads to:

Property 10 (original series decomposition)
A series $S(p,q)$ with the period q odd can be decomposed as subsetting of itself of given evenness, the subsetting series of evenness i having a period 2^{i+1}:
$$S(p,q)=\bigcup_{i\in N} S(p,q)//S(x_i,2^{i+1})$$
It is linked with the subsetting of the odd numbers with explicit evenness (*property 9*) by the relation:
$$x_i=\left(-p+2^i+2^{i+1}*a_i\right)/q \text{ .}$$

Knowing these periods have a simple consequence on their organisation. If we know an index k of the original series such that the number has an evenness of i, then we can find the generator of this series by removing the period until the last positive integer. This gives simply $g_i=k-2^{i+1}*E\left[k/2^{i+1}\right]$, where E stands for the integer part ($E\left[k/2^{i+1}\right]$ being precisely the largest possible integer p such that $0\le k-p*2^{i+1}$). We also know that the two surrounding numbers $k+1$ and $k-1$ have a null evenness, as it is the only possibility left with a period 2. Again, $k+2$ and $k-2$ will have an evenness of 1 (only possibility for a period 2^2), $k+3$ and $k-3$ will be back to evenness 0, $k+4$ and $k-4$ will have an evenness of 2 (only possibility for a period 2^3), and so on and so forth. In general we know that $k+2^j$ and $k-2^j$ will have an evenness of j. With all these series we can find their respective generators. We can do so until we reach $j=i$. Then we are left with the numbers $k\pm 2^i$, which we do not know a priori the evenness, except that it is strictly larger than i. If we have for instance $k+2^i$ with an evenness l, then we know again all the surrounding numbers until the same evenness l, including that $k-2^i$ has an evenness of $i+1$.
We can summarize this by:

Property 11 (filling property)
For a series $S(p,q)$ with q odd, knowing an index k of evenness i ($e(p+k*q)=i$), imposes all the generators of the subsetting series of evenness $j\le i$, $S(p,q)//S(x_j,2^{j+1})$ with:
$$x_j=k+2^j-2^{j+1}*E\left[(k+2^{j+1})/2^{j+1}\right] \text{ .}$$

For a given series $S(p,q)$ we can characterise the succession of evenness by the succession of the generators x_i (*property 10*), but also a_i (*property 9*). In general we do not know what this succession will be, however we can see that the explicit evenness decomposition keeps for the new series to be decomposed $S(p^+,q)$ *the same period as the original one* (q), while changing the generator. To each generator of series to be decomposed p correspond only one decomposing series generator a, so it is equivalent to consider p or a. As each new series is always full (*property 6*), the generator a is smaller than q. So the number of possible generator is at most q, so it is bound to pass again by the same generator after at most q iterations. If it comes back on the same generator, then it will repeat the same odd/even decomposition series, as it depends only on the last value and the period q (*property 8*), and so on periodically. Thus :

Property 12 ("a" decomposition periodicity)

A series $S(p,q)$ with the period q odd decomposes on all the possible evenness i with periodic odd subsetting generators a_i, with a period at most q.

These last four properties define the same perfectly regular fractal structures (with just different a_i series) that we always find when decomposing a series with odd period. The general question that remains is about these periodic generator series a_i. They will depend on the particular values of p and q, and the previous formula of *property 7* or *8* allows to compute them recursively.

It is interesting to combine the last properties, namely the properties coming form the a decomposition and the properties coming from the x one. For instance we know the a decomposition is periodic (*property 12*), and we also know the decomposing periods (*property 9*). So if the largest know evenness is larger than the maximum possible period, we know we have a full period known (with *property 10*), and we can deduce all the decomposition. Even if it is lower, in case the decomposition already contains a period, we also knowoç the rest of it. If we do not find a period, we cannot deduce the position (phase) of the higher evenness because we do not know explicitly the period of the decomposition (we just know its maximum possible value).

We can still find some other general properties. For a given period q, the generator is limited between the two extreme case 0 and $q-1$. For a series $S(p,q)$ we can thus look at the "symmetric" series $S(\overline{p},q)$, with generator $\overline{p}=q-p$ instead of p. We have for its decomposition:

-•- if p is odd ($p=1+2p'$) :

$S(q-p,q)//S(1,2)=S(-1+2q-2p',2q)=S(1,2)//S(q-1-p',q)$ while

$S(q-p,q)//S(0,2)=S(2q'-2p',2q)=2*S(q'-p',q)$,

-•- if p is even ($p=2p'$):

$S(q-p,q)//S(0,2)=S(1+2q'-2p',2q)=S(1,2)//S(q'-p',q)$ while

$S(q-p,q)//S(1,2)=S(2q-2p',2q)=2*S(q-p',q)$,

which can be summarized as above by:

$$\left\langle \begin{array}{ll} p=1+2p': & \overline{a}=q-1-(p'), \qquad \overline{p}^+=q'-p'=q-(1+p'+q') \\ p=2p': & \overline{a}=q'-p'=q-1-(p'+q'), \quad \overline{p}^+=q-(p') \end{array} \right.$$

We thus observe a global symmetry:

Property 13 ("a" central symmetry)
A series $S(p,q)$ and its "symmetric" $S(\overline{p},q)$ with $\overline{p}=q-p$, gives "symmetric" first odd subset generators $\overline{a}=q-1-a$ and next generator $\overline{p}^{+}=q-p^{+}$. Thus the two periodic series a and $\overline{a}$ are "symmetric" from each other.

So if we look, for a given period q, at all the a series that we can find with all the possible starting generators p , we can see that they are globally symmetric, either one to another, or one with itself. In this last case, the number of iteration of the decomposition to go from one value a to its symmetric value $\overline{a}=q-1-a$ is exactly the same than to go from the value $\overline{a}$ to a :

Property 14 (possible "a" series symmetry)
If an a series is symmetric with itself, and we call r the a repetition period ($a(i+r)=a(i)$), we thus know that r is even, $r=2\,r'$, and that $a(i+r')=\overline{a}(i)=q-1-a(i)$.

As an example we can draw in *figure 4* the a series of the decompositions of all the possible series with odd period 15. We can see self-symmetric trajectories, as well as trajectories symmetric with some others.

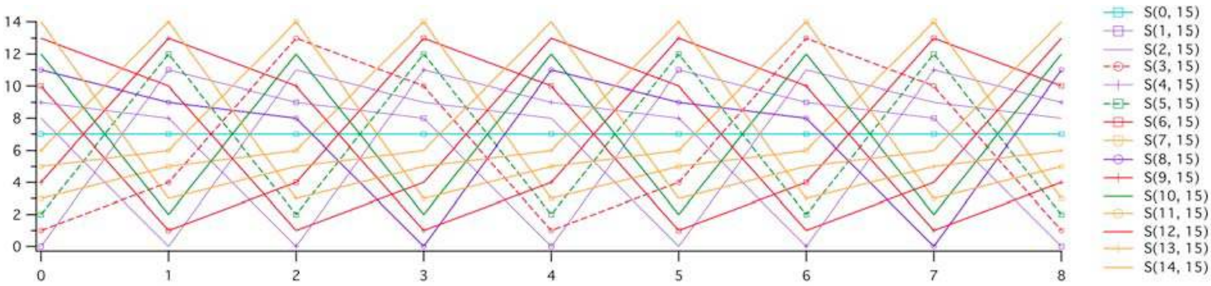

Fig. 4-"a" series for q=15. In total we find 1 *trajectory of period* 1 ($p=0$)*,* 2 *of period* 2 ($p\in\{5,10\}=5*\{1,2\}$)*,* 4 *of period* 4 ($p\in\{3,6,9,12\}=3*\{1,2,3,4\}$) *(all of them self-symmetric), and* 4 *of period* 4 ($p\in\{1,2,4,8\}$) *symmetric with some other 4* ($p\in\{14,13,11,7\}=15-\{1,2,4,8\}$).

In general we can find some other properties in the group of the q full series with a given period q. Each a_i value in a trajectory corresponds in the decomposition to a series $S(p_i,q)$ with the same period q so also member of the group. At each step this decomposing series can thus be taken as a new original series from the group. After, they will share the same periodic decomposition (trajectory). If this trajectory is of period r, it contains r different values (as the new value depending only on the preceding one, it cannot be equal to a previous one except at a period), and it thus correspond to r identical trajectories except shifted by one step.
Similarly, for all the other trajectories, the number of values is equal to the periodicity and number of identical trajectories shifted by one step. Globally all these trajectories take all the possible q values. Thus:

Property 15 (global "a" trajectories)
In the group of decomposition of all the series with same period q, if there is a trajectory with period r, there will be the same trajectory repeated r times (in total) just shifted by one in indexes.
The sum of the periods of all the types of different (not identical by translation) trajectory is equals to q.

We can check this in the previous example (*fig. 4*), with 15=1+2+4+2*4.

In the decomposition for prime periods q, we find the decomposition with $q*S(0,1)$ having the same 1 period decomposition than $S(0,1)$, and often $q-1$ (even) shifted self-symmetric trajectories of period $q-1$, but we can also find a group of two asymmetrical trajectories (symmetric with each other), with a $(q-1)/2$ period (like for q=7, 17, 23…).

In general, if the period q is not prime, for instance $q=s*r$, then among all the possible generators p we will have the multiples of r and of s. For instance, we will have $p=p'*r$, and $S(p,q)=r*S(p',s)$. We can decompose this series of period s, finding its own trajectories, and multiply it back by r. For each evenness i it will give:
$r*2^i*S(1,2)//S(a_i,s)=2^i*S(1,2)//S((r-1)/2+r*a_i,q)$
so we will find the same trajectory, but linearly modified following:
$a_i(p,q)=a_i(p'*r,s*r)=(r-1)/2+r*a_i(p',s)$

In particular this tells that the first possible series, $S(0,q)=q*S(0,1)$ is a constant trajectory with a value $(q-1)/2$, which is the symmetry line. If we rewrite each trajectory around this symmetry line, as $\tilde{a}_i=a_i-(q-1)/2$, the previous transformation can be more simply rewritten as $\tilde{a}_i(p,q)=r*\tilde{a}_i(p',s)$. In summary we have:

Property 16 ("a" trajectories of multiple series)
When a series is a multiple of another one, $S(p,q)=r*S(p',s)$, then the a series of the multiple can be deduced from the first one by $a_i(p,q)=(r-1)/2+r*a_i(p',s)$, and if centred around the central value $a=(q-1)/2$ (obtained for $S(0,q)=q*S(0,1)$), by $\tilde{a}_i=a_i-(q-1)/2$, then simply $\tilde{a}_i(p,q)=r*\tilde{a}_i(p',s)$.

For instance in the above $q=15$ example of *fig. 4* we have the decomposition of period 3, multiplied by 5 ($5*S(1,3)$ and $5*S(2,3)$) as well as of period 5, multiplied by 3 (from $3*S(1,5)$ to $3*S(4,5)$).

So if we look at all the multiples of a period q, it will gives these trajectories derived from previous trajectories of smaller period, and then some other globally symmetric trajectories to fill the rest, with maximum possible periodicity equal to the number of left possible values. Again, we find in general either self-symmetric as for q = 9, 25, 27, …, or two type of trajectories symmetric with each other, as for q = 15, 21…

Property 17 ("a" trajectories of series with a period power of prime)
We can look at a particular case we found later, when the period is a power of a prime number, $q=p^n$. Then the a trajectories will contain the trajectories of
- $p^n*S(0,1)$, giving the symmetry line.
- $p^{n-1}*S(g,p)$ with all the possible g except the preceding one, so $1\le g<p$, ($p-1$ values)
- $p^{n-2}*S(g,p^2)$ similarly for all the possible g except the preceding ones, so for g relatively prime with p^2 (which leaves p^2-p values)
- …
- $p^{n-m}*S(g,p^m)$ for g relatively prime with p^m (which leaves p^m-p^{m-1} values)
…
- until $S(g,p^n)$ for g relatively prime with p^n (which leaves p^n-p^{n-1} values).
For each limited number of possible g values, this lead to a corresponding maximum possible a period, as the series can take only the corresponding a values.

As for the previous cases, it seems that we always find self-symmetric trajectories with the maximum possible period, as we saw for 3, 9, 27, or 5, 25 ; or two respectively symmetric trajectories with half the maximum possible period, as for 7 and 49. *Is it always the case, and what is the repartition between the two cases?*

As a first conclusion we can see that the decomposing series are quite structured, and even if we restrict to the fractal structure (*property 11*), we find only particular periodic cases (*property 12*), and apparently with even more properties as we seem to find only the self-symmetric trajectories with maximum period or two symmetric trajectories with half the maximum period.
More generally, combining this a iteration of *property 9*, with the x derivation of *property 10* seems promising, as it imposes many constraints, and thus should reveal many properties.

This decompostion procedure can also be generalized to other numbers than 2, and thus define a “n-ity” (the value of the number modulo n), and a “n-ity decomposition”. For instance, if we develop on the number 3, we can define a “thirdity” and a “thirdity decomposition”. It should decompose a series along the possible thirdity value, equivalent to an “a” decomposition, but we can obtain it by decomposing the original series along the thirdity of the index, equivalent to an “x” decomposition.

With these tools we can now look at the iterations of C.

5 - The first Iteration

We start with the odd numbers $S(1,2)$ and apply T to it:
$T(S(1,2)) = S(3*1+1, 3*2) = S(4, 3*2) = 2*S(2,3)$.
We can see that once divided by 2 we are in the odd period case, with a prime period (3), so we know it will decompose indefinitely (*property 9*) with subsetting generator period at most 3 (*property 12*). Knowing that there is necessarily the constant a series corresponding to $3*S(0,1)$, the rest will consist either of two symmetric trajectories of period 1 (thus constant) or one self symmetric series of period 2 (*property 15*) . We could compute it directly using *property 8*, but let us compute it explicitly:
$S(2,3) = S(2,3)//S(0,2) \cup S(2,3)//S(1,2)$
now, as p is even and q odd, we have:
$S(2,3)//S(0,2) = S(2, 2*3) = 2*S(1,3)$ is a series of even numbers and on the contrary $S(2,3)//S(1,2) = S(5, 3*2) = S(1,2)//S(2,3)$ (a case of particular commutation between maximum generator full series, *property 1*) is a subset of the odd numbers. The next decomposition is
$S(1,3) = S(1,3)//S(0,2) \cup S(1,3)//S(1,2)$
with now, as p and q are odd:
$S(1,3)//S(0,2) = S(1, 3*2) = S(1,2)//S(0,3)$ is the subset of the odd numbers, while $S(1,3)//S(1,2) = S(4, 3*2) = 2*S(2,3)$ is the even part. But we are back to the original series, so we can write :
$S(2,3) = 2^2*S(2,3) \cup 2*S(1,2)//S(0,3) \cup S(1,2)//S(2,3)$
Repeating it to infinity we can write:

$$S(2,3) = \bigcup_{i \in N} 2^{2i+1} * S(1,2) // S(0,3) \bigcup_{i \in N} 2^{2i} * S(1,2) // S(2,3)$$

or

$$T(S(1,2)) = \bigcup_{i \in N} 2^{2i+2} * S(1,2) // S(0,3) \bigcup_{i \in N} 2^{2i+1} * S(1,2) // S(2,3)$$

Following the form of *property 9*, decomposition on the odd numbers, $S(2,3) = \bigcup_{i \in N} 2^i * S(1,2) // S(a_i, 3)$, we find among the three possible values ($0 \le a(i) < 3$) a repetition of a of period 2 (necessarily smaller than $q = 3$ from *property 12*) : $a(i) = \{\underline{0,2},...\}$. This trajectory is self-symmetric, as it passes on 0 and its symmetric value $q-1=2$ (*property 14*). This leaves the last possible case, $a=1$, which corresponds to $S(1,2)//S(1,3) = S(3, 3*2) = 3*S(1,2)$, the series of the odd multiples of 3, absent in the iterations (as is well known). This last series correspond to an a decomposition of period 1, as the N decomposition (*property 16*) :

$$3*N = S(0,3) = \{0\} \bigcup_{i \in N} 2^i * 3 * S(1,2) = \{0\} \bigcup_{i \in N} 2^i * S(1,2) // S(1,3).$$

The result of d for the first iteration has exactly the same structure than e (*property 9*), except that (thanks to T) all the values are increased by 1, and that now the generators and the iterations of the evenness are shifted (because we have not only $a=0$, but also periodically $a=2$):

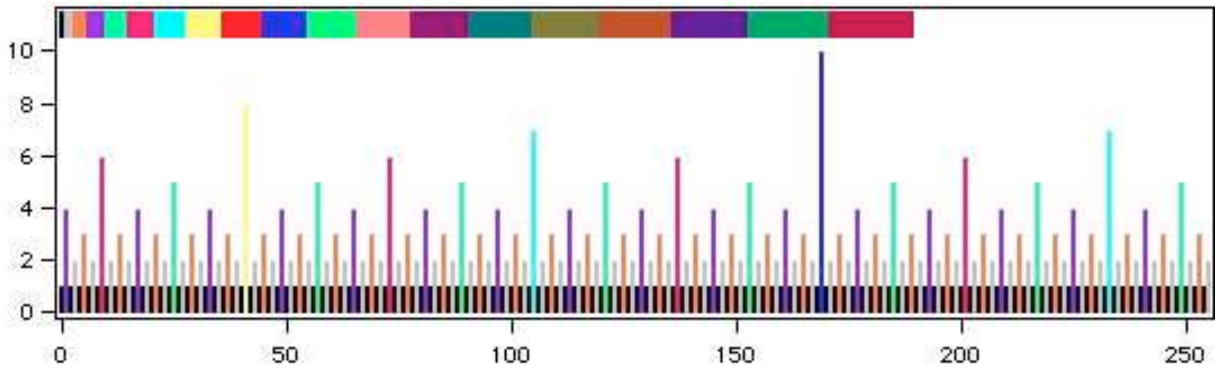

Fig 5 – First history of odd numbers, with the same coding as fig. 1. The minimum value is 1, and we find the same fractal structure as the evenness (fig. 2) except the values are shifted (compare for instance 8=yellow, that now comes before 7=light blue). This also corresponds to the lower pixels of fig. 1.

We can also rewrite this first iteration image decomposition as subsets of the original one (*property 10*) :

$$S(2,3) = \bigcup_{i \in N} S(2,3) // S(x_i, 2^{i+1})$$

with

$$x_i = \left(-2 + 2^i + 2^{i+1} * a_i\right)/3$$

Using the result $a(i) = \{\underline{0,2},...\}$, it gives for $i = 2j+1$ odd :

$$x_{2j+1} = 2*\left(-1+2^{2j}\right)/3,$$

and for $i = 2j$ even :

$$x_{2j} = 2*\left(-1+5*2^{2j-1}\right)/3$$

It is not obvious at first sight that these subset generators are integers, but we know they are, by construction. The beginning of the series reads:

$\{1, 0, 6, 2, 26, 10, 106, 42, 426, 170, 1706, 682, 6826, 2730, 27306, 10922, ...\}$

Recalling that $T(S(1,2)) = 2 * S(2,3)$, and equalling the two writings, we can write :
$T(S(1,2)) = \bigcup_{i \in N} T(S(1,2)) // S(x_i, 2^{i+1}) = \bigcup_{i \in N} 2^{i+1} * S(1,2)) // S(a_i, 3)$
As the function T is simply arithmetic, $T(S(p,q)) // S(r,s) = T(S(p,q) // S(r,s))$
(*property 2*), we can write for each i :
$T\left(S(1,2) // S(x_i, 2^{i+1})\right) = 2^{i+1} * S(1,2)) // S(a_i, 3)$
so this decomposition gives the original subsets that will transform with an iterate evenness $d = i + 1$, and the results:
$C(S(1,2) // S(x_i, 2^{i+1})) = S(1,2) // S(a_i, 3)$
Explicitly this gives
for $i = 2j + 1$ odd :
$C(S(1,2) // S(x_i, 2^{i+1})) = S(1,2) // S(2,3) = S(5, 2*3)$
and for $i = 2j$ even :
$C(S(1,2) // S(x_i, 2^{i+1})) = S(1,2) // S(0,3) = S(1, 2*3)$

In other words we know all the numbers that have an iterate evenness $d = i$, that we write $\overset{1}{H} = h_1 = \{i\}$, are the full periodic series $G(h_1) = S(1,2) // S(x_{i-1}, 2^i)$, and the result by C will be $C(G(h_1)) = S(1,2) // S(a_{i-1}, 3)$.

6 - The second Iteration

The next step will be to apply C to the two result series $S(1, 3*2) = S(1,2) // S(0,3)$ and $S(5, 3*2) = S(1,2) // S(2,3)$.

Let us first look at
$T(S(5, 3*2)) = S(16, 3^2 * 2) = 2 * S(8, 3^2)$
Again, we could filter this image with the decomposition of N, or even filter the original series with the previous decomposition. We could also use the previous tools, as we are again in the case of an odd period (*properties 7 to 16*), even a power of a prime number $q = 3^2$ (*property 17*). But to show again an explicit example we will decompose it directly. The first odd/even decomposition gives:
$S(8, 3^2) = S(8, 3^2) // S(0,2) \cup S(8, 3^2) // S(1,2)$ with
$S(8, 3^2) // S(0,2) = S(8, 3^2 * 2) = 2 * S(4, 3^2)$ are even numbers while
$S(8, 3^2) // S(1,2) = S(17, 3^2 * 2) = S(1,2) // S(8, 3^2)$ are odd numbers.
The process can be repeated the same way easily, and we get this time a period 6:

$$S(8, 3^2) = 2^6 * S(8, 3^2) \cup 2^5 * S(1,2) // S(3, 3^2) \cup 2^4 * S(1,2) // S(2, 3^2) \cup 2^3 * S(1,2) // S(0, 3^2)$$
$$\cup\ 2^2 * S(1,2) // S(5, 3^2) \cup 2^1 * S(1,2) // S(6, 3^2) \cup 2^0 * S(1,2) // S(8, 3^2)$$

which gives:

$$T(S(5, 3*2)) = \bigcup_{i \in N} 2^{6i+6} * S(1,2) // S(3, 3^2) \bigcup_{i \in N} 2^{6i+5} * S(1,2) // S(2, 3^2) \bigcup_{i \in N} 2^{6i+4} * S(1,2) // S(0, 3^2)$$
$$\bigcup_{i \in N} 2^{6i+3} * S(1,2) // S(5, 3^2) \bigcup_{i \in N} 2^{6i+2} * S(1,2) // S(6, 3^2) \bigcup_{i \in N} 2^{6i+1} * S(1,2) // S(8, 3^2)$$

So now with a subsetting generators series $a_2(i) = \{\underline{8, 6, 5, 0, 2, 3}, ...\}$:

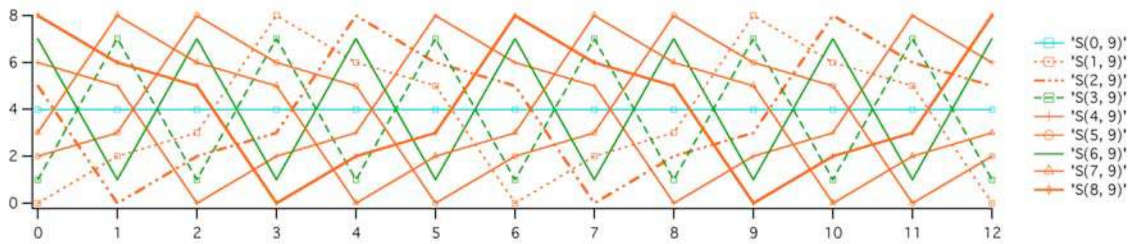

*Fig. 6 – The "a" series corresponding to the second iteration, among all the possible with q=9. The first series (thick orange) is for 2*S(8, 9)=T(S(5,6)), while the long-dot-dot dashed one is for the other series 2*S(2, 9)=T(S(1,6)).*

We find a periodic decomposition, symmetric, with period 6 (smaller than 3^2). It contains all the possible subsets of the odd numbers with a period of 3^2, $S(1,2)//S(a,3^2)$, except the one corresponding to multiples of 3 :

$S(1,2)//S(1,3^2) = 3*S(1,3*2) = 3*S(1,2)//S(0,3)$,

$S(1,2)//S(4,3^2) = 3*S(3,3*2) = 3^2*S(1,2)$, and

$S(1,2)//S(7,3^2) = 3*S(5,3*2) = 3*S(1,2)//S(2,3)$).

Each subsetting series $S(1,2)//S(a,3^2)$ comes from the decomposition of an original full series, $S(p,3^2)$. We can see that the multiple of 3^2 comes from the period 1 the decomposition of $3^2 N$:

$$3^2 * N = S(0,3^2) = \{0\} \bigcup_{i \in N} 2^i * 3^2 * S(1,2) = \{0\} \bigcup_{i \in N} 2^i * S(1,2)//S((3^2-1)/2, 3^2) = \{0\} \bigcup_{i \in N} 2^i * S(1,2)//S(4,3^2)$$

while the two others multiples of 3 come from the previous period 2 symmetric decomposition of the first iteration:

$$\begin{aligned} S(6,3^2) = 3*S(2,3) &= \bigcup_{i \in N} 2^{2i+1} * 3 * S(1,2)//S(0,3) \bigcup_{i \in N} 2^{2i} * 3 * S(1,2)//S(2,3) \\ &= \bigcup_{i \in N} 2^{2i+1} * S(1,2)//S(1,3^2) \bigcup_{i \in N} 2^{2i} * S(1,2)//S(7,3^2) \end{aligned}$$

and $S(3,3^2) = 3*S(1,3)$ being the same decomposition shifted by one step. The removal of the multiples of 3, of period 1 and 2, explains why the periodicity of the decomposition is only $2*3 = 3^2 - (2*3^0 + 1)$.

Again this writing shows that we have the same fractal structure, except with different (periodic) origins (property 9):

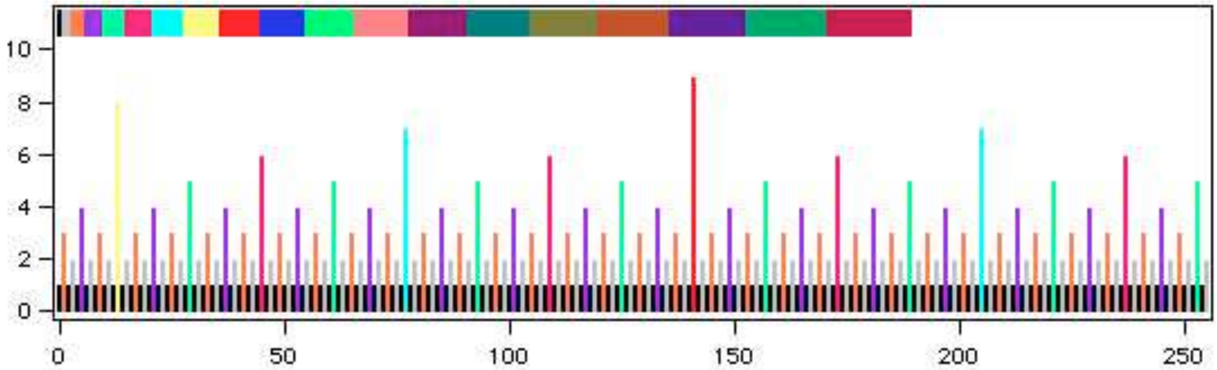

Fig 7 – Second history of the first resulting odd numbers, S(5, 6), with the same coding as fig. 1. As in fig. 5, the minimum value is 1, and we find the same fractal structure as the evenness (fig. 2) except the values are again shifted (8=yellow now even comes before 6=pink and 5=turquoise).

We can also draw this periodic repetition of the decomposition in a schematic way:

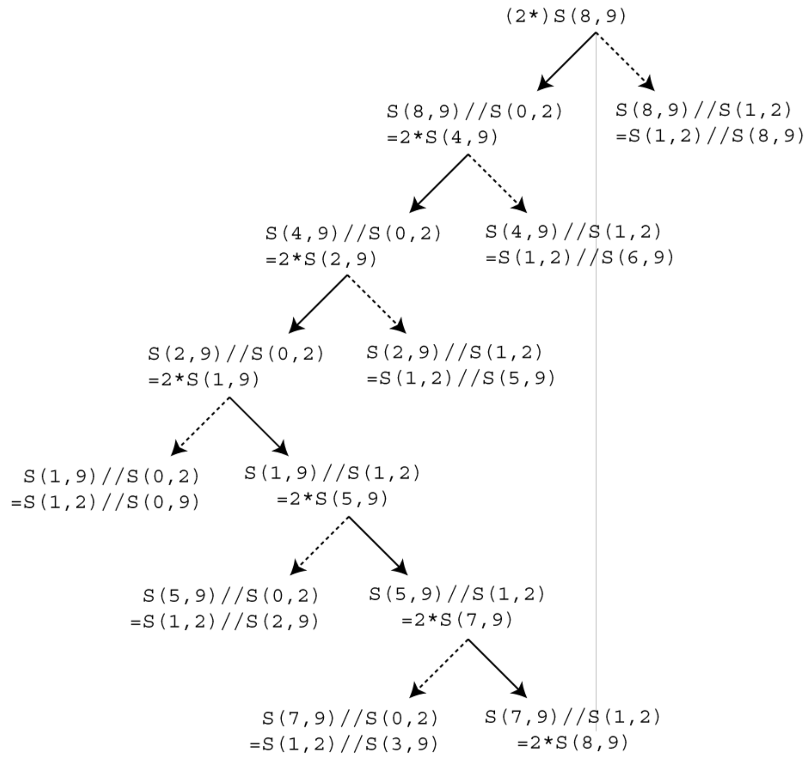

Fig 8 – Sketch of the odd/even decomposition of S(8, 9).

The fact that the decomposition of $S(8,3^2)$ goes through all the possible *a* values (except the one corresponding to multiples of 3), means that the decomposition of the original series $S(p,3^2)$ (except for *p* multiple of 3) will give the same decomposition, except at shifted by some value. So if we look at the iteration of the other result series:
$T(S(1,3*2)) = S(4,3^2*2) = 2*S(2,3^2)$,
we know right away that it is the same decomposition, except shifted. Looking at the picture tells us that here it is shifted two steps ahead. So will find the same periodic series except with a difference phase in the evenness:

$$T(S(1,3*2)) = \bigcup_{i\in N} 2^{6i+6}*S(1,2)//S(6,3^2) \bigcup_{i\in N} 2^{6i+5}*S(1,2)//S(8,3^2) \bigcup_{i\in N} 2^{6i+4}*S(1,2)//S(3,3^2)$$
$$\bigcup_{i\in N} 2^{6i+3}*S(1,2)//S(2,3^2) \bigcup_{i\in N} 2^{6i+2}*S(1,2)//S(0,3^2) \bigcup_{i\in N} 2^{6i+1}*S(1,2)//S(5,3^2)$$

We can summarize both the periodicity of the decomposition and the image of the result series into the next ones, by the following drawing where we indicated only the period (on top), the generator of the odd subset (understating that they repeat periodically), and its image in the next iteration (drawing the original series as $S(1,2) = S(1,2)//S(0,1)$ so an odd subset generator 0):

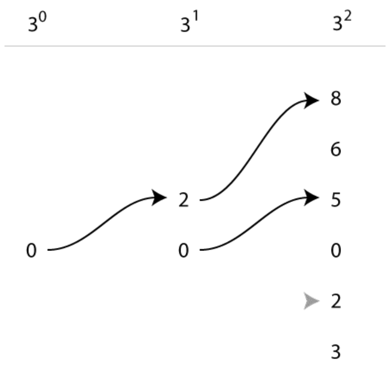

Fig. 9 – Summary of the first two iterations.

This diagram contains all the information for the first two iterations. For instance, following the idea of Terras (1976), we can look at the numbers that have the same beginning of history $h_2 = \{i, j\}$ (with $1 \le i$ and $1 \le j$). The first step is to take in the image $T(S(1,2))$ the subset $2^i S(1,2) // S(a(1,i), 3^1)$. To recover the original series we have to write it as subset of the image $T(S(1,2)) = 2 * S(2,3)$:

$2^i S(1,2) // S(a(1,i), 3^1) = 2 * S(2^{i-1}(1+2a(1,i)), 2^i * 3^1) = 2 * S(2,3) // S(x, 2^{i+1}) = 2 * S(2+3x, 2^i * 3^1)$

so

$2 + 3^1 x = 2^{i-1}(1 + 2a_1(i))$

or finally

$x_1(i) = \left(-2 + 2^{i-1}(1 + 2a_1(i))\right) / 3^1$

so the original subset is $G(i) = S(1,2) // S(x_1(i), 2^i) = S(1 + 2 * x_1(i), 2^{i+1})$

with a period $q_1(i) = 2^{i+1}$ and a generator $g_1(i) = 1 + 2 * x_1(i)$

and the result is $C(G(i)) = S(1,2) // S(a_1(i), 3^1)$.

Now the difficulty for the next subsetting, giving among this result the numbers having the right next evenness, is that, although the series is the same for the decomposition of both results, the phase is different, as indicated by the two arrows. We thus introduce the phase $\varphi(1,i)$ depending on the previous starting point $a(1,i)$ ($\varphi_1(i) = 0$ for i odd and $\varphi_1(i) = 2$ for i even, in other words $\varphi(1,i) = \left\{\underline{0,2},...\right\}$ with $1 \le i$).

So the right subset of $T(C(G(i)))$can be written as $2^j S(1,2) // S(a(2, j + \varphi_1(i)), 3^2)$.

Similarly we have to rewrite it as a subset of the result $2 * S(2,3) // S(a(1,i), 3^1)$:

$2^j S(1,2) // S(a(2, j+\varphi_1(i)), 3^2) = 2 * S(2^{j-1}(1 + 2a(2, j+\varphi_1(i))), 2^j * 3^2)$

$= 2 * S(2,3) // S(a(1,i), 3^1) // S(x, 2^j) = 2 * S(2 + 3^1 * a(1,i) + 3^2 * x, 2^j * 3^2)$

so

$2 + 3 * a(1,i) + 3^2 * x = 2^{j-1}(1 + 2a(2, j + \varphi_1(i)))$

and finally

$x_2(i,j) = \left[-2 - 3 * a(1,i) + 2^{j-1}(1 + 2a(2, j + \varphi_1(i)))\right] / 3^2$.

To find the original series, we just have to recall that this subsetting is a subsetting of the previous one.

Thus, from the diagram [$a_n(i)$ and $\varphi_n(i)$] we are able to compute explicitly the series of numbers with any first two steps in history $h_2 = \{i, j\}$ (with $1 \le i$ and $1 \le j$):

$G(i,j) = S(1,2) // S(x_1(i), 2^i) // S(x_2(i,j), 2^j) = S(1 + 2x_1(i) + 2^{i+1} x_2(i,j), 2^{1+i+j})$ with

$x_1(i) = \left(-2 + 2^{i-1}(1 + 2a_1(i))\right)/3^1$ and

$x_2(i,j) = \left[-2 - 3^1 * a_1(i) + 2^{j-1}(1 + 2a_2(j + \varphi_1(i)))\right]/3^2$,

so a series $S(1 + 2x_1(i) + 2^{i+1}x_2(i,j), 2^{1+i+j})$

and period $q_2(i,j) = 2^{1+i+j}$

and a generator $g_2(i,j) = 1 + 2x_1(i) + 2^{i+1}x_2(i,j)$.

When expanded, the generator can be rewritten into

$g_2(i,j) = -(1/3)(1 + 2^i/3) + 2^{i+j}/3^2 + 2^{1+i+j}a_2(j + \varphi_1(i))/3^2$

The result after the first iteration will be

$C(G(i,j)) = S(1,2) // S(a_1(i),3) // S(x_2(i,j), 2^j)$

and after the second iteration:

$C^2(G(i,j)) = S(1,2) // S(a_2(j + \varphi_1(i)), 3^2)$.

We can see that the period in the origin series is rather simple, $q_2(i,j) = 2^{1+i+j}$, but the generator $g_2(i,j)$ is more complex. For the first iteration we have:

$x_1(i) = \{1, 0, 6, 2, 26, 10, 106, 42, 426, 170, 1706, 682, 6826, 2730, 27306, 10922, ...\}$

which gives the generators $g_1(i) = 1 + 2x_1(i)$:

$g_1 = \{3, 1, 13, 5, 53, 21, 213, 85, 853, 341, 3413, 1365, 13653, 5461, 54613, 21845, ...\}$

For the next iteration, we have, for *i* odd:

$x_2(1,j) = \{1, 2, 4, 0, 8, 24, 120, 184, 312, 56, 568, 15932, 7736, 11832, 20024, 3640, ...\}$

and for *i* even:

$x_2(2,j) = \{1, 0, 2, 6, 30, 46, 78, 14, 142, 398, 1934, 2958, 5006, 910, 9102, 25486, ...\}$

which gives the table (up to 16):

i\j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	**7**	**11**	19	3	35	99	483	739	1251	227	2275	6371	30947	47331	80099	14563
2	9	**1**	**17**	49	241	369	625	113	1137	3185	15473	23665	40049	7281	72817	203889
3	**29**	45	77	**13**	141	397	1933	2957	5005	909	9101	25485	123789	189325	320397	58253
4	**37**	**5**	69	197	965	1477	2501	453	4549	12741	61893	94661	160197	29125	291269	815557
5	117	**181**	309	**53**	565	1589	7733	11829	20021	3637	36405	101941	495157	757301	1281589	233013
6	**149**	21	**277**	789	3861	5909	10005	1813	18197	50965	247573	378645	640789	116501	1165077	3262229
7	**469**	**725**	1237	213	2261	6357	30933	47317	80085	14549	145621	407765	1980629	3029205	5126357	932053
8	597	**85**	**1109**	3157	15445	23637	40021	7253	72789	203861	990293	1514581	2563157	466005	4660309	13048917
9	**1877**	2901	4949	**853**	9045	25429	123733	189269	320341	58197	582485	1631061	7922517	12116821	20505429	3728213
10	**2389**	**341**	4437	12629	61781	94549	160085	29013	291157	815445	3961173	6058325	10252629	1864021	18641237	52195669
11	7509	**11605**	19797	**3413**	36181	101717	494933	757077	1281365	232789	2329941	6524245	31690069	48467285	82021717	14912853
12	**9557**	1365	**17749**	50517	247125	378197	640341	116053	1164629	3261781	15844693	24233301	41010517	7456085	74564949	208782677
13	**30037**	**46421**	79189	13653	144725	406869	1979733	3028309	5125461	931157	9319765	26096981	126760277	193869141	328086869	59651413
14	38229	**5461**	**70997**	202069	988501	1512789	2561365	464213	4658517	13047125	63378773	96933205	164042069	29824341	298259797	835130709
15	**120149**	185685	316757	**54613**	578901	1627477	7918933	12113237	20501845	3724629	37279061	104387925	507041109	775476565	1312347477	238605653
16	**152917**	**21845**	283989	808277	3954005	6051157	10245461	1856853	18634069	52188501	253515093	387732821	656168277	119297365	1193039189	3340522837

Fig. 10 – Table of the generators $g_2(i,j)$ of the series of numbers with first history $\{i,j\}$, with $i, j \in \{1, 16\}$.

We can see that the generators values increase globally. Following the previous formula giving $g_2(i,j)$ depending on a_2 we can use that $0 \le a_2 \le 3^2 - 1$ in order to obtain the minimum and maximum values for $g_2(i,j)$:

$(-3 - 2^i + 2^{i+j})/3^2 \le g_2(i,j) \le (-3 - 2^i + 17 * 2^{i+j})/3^2$

We find that the minimum values are reached for

$i \in S(1,2), j \in S(4,6)$ or $i \in S(0,2), j \in S(2,6)$, which correspond indeed to the two possible ways to reach $a_2 = 0$ in the graph of *fig. 8* (recalling that $a_1 = 2$ correspond to *i* odd, $i \in S(1,2)$, $a_1 = 0$ to $i \in S(0,2)$, $a_2 = 8$ to $j \in S(1,6)$, $a_2 = 6$ to $j \in S(2,6)$ etc.).

Similarly, the maximum values are reached for

$i \in S(1,2), j \in S(1,6)$ or $i \in S(0,2), j \in S(5,6)$ ($a_2 = 8$).

This table is also highly structured. One can first notice that the iteration of these generators only generates… generators within the same table. For instance

$$H\overset{\infty}{/C}(7509) = \{7509, \frac{11}{11}, \frac{1}{17}, \frac{2}{13}, \frac{3}{5}, \frac{4}{1}, \frac{2}{\underline{1}}, ...\}$$

With this writing we can see that the first history of 7509 is {11, 1}, and we can check that it is a generator of this history as $7509 < 2^{1+11+1} = 8192$. Similarly $\overset{2}{H}(11) = \{1,2\}$ is also a generator as $11 < 2^{1+1+2} = 16$, and so on and so forth. Other examples can be

$$H\overset{\infty}{/}C(203861) = \{203861, \frac{8}{2398}, \frac{10}{11}, \frac{1}{17}, \frac{2}{13}, \frac{3}{5}, \frac{4}{1}, \frac{2}{\underline{1}}, ...\}$$ or

$$H\overset{\infty}{/}C(1514581) = \{1514581, \frac{8}{17749}, \frac{12}{13}, \frac{3}{5}, \frac{4}{1}, \frac{2}{\underline{1}}, ...\}$$ or

$$H\overset{\infty}{/}C(1631061) = \{1631061, \frac{9}{9557}, \frac{12}{7}, \frac{1}{11}, \frac{1}{17}, \frac{2}{13}, \frac{3}{5}, \frac{4}{1}, \frac{2}{\underline{1}}, ...\}$$ …

The iterations are also structured in a particular way: each column *j* is iterated on only two first elements (with *j*≤4) of the line *j* (indicated in bold and coloured background). For instance the column *j*=1 is iterated into ***11*** and *7* periodically, the column *j*=2 into ***17*** and ***1***, *j*=3 into **29** and ***13***, etc… The pattern of the image elements is periodic of period 6 (indicated by the bordering frame). This is shown explicitely in *Appendix A*.

This shows that in the first iteration all the generators are projected into the first four columns, and then they are projected, in the second iteration, into the italic-bold generators in the first 4*4 corner (namely *7*, ***11***, ***1***, ***17***, **29**, ***13***, **37** and ***5***). In fact we know that they will converge toward only 6 of them (italics, coloured in light green), as we know that the result from the first two iterations will be $S(1,2)//S(a_2, 3^2)$, where a_2 takes only six values : $a_2 = \{8,6,5,0,2,3\}$, corresponding to only six possible generators $x_2 = 1 + 2a_2 = \{17,13,11,1,5,7\}$.

It is easy to check that the next iterations of these generators also quickly winds down to… the fixed point $\overset{\infty}{C}(1) = \{1\underline{\{2,2\}}, ...\}$: $\overset{\infty}{C}(37) = \{37, 7, 11, 17, 13, 5, \underline{1}, ...\}$ and $\overset{\infty}{C}(29) = \{29,11,17,13,5,\underline{1}, ...\}$. Thus all the second iteration generators $g_2(i, j)$ converge to the fixed point 1 in at most 8 iterations (7 in fact, as only the column j=3 and j=4 for i odd tranfer to 29 and 37, and that all the other iterations on these two columns happen for even number…). This nice result of course does not apply to all the other numbers which are not second order generators.

Note also that the first iteration generators $g_1(i) = 1 + 2x_1(i)$ are included in this table, as $g_1(iodd) = g_2(i,4)$ for *i* odd and $g_1(ieven) = g_2(i,2)$ for *i* even (as $a(2,4) = 0$). They converge toward 1 even more quickly (in 2 steps for *i* odd, 1 for *i* even).

7 - The third and fourth Iterations

We can follow the same procedure as above, and summarize it into a more complex diagram:

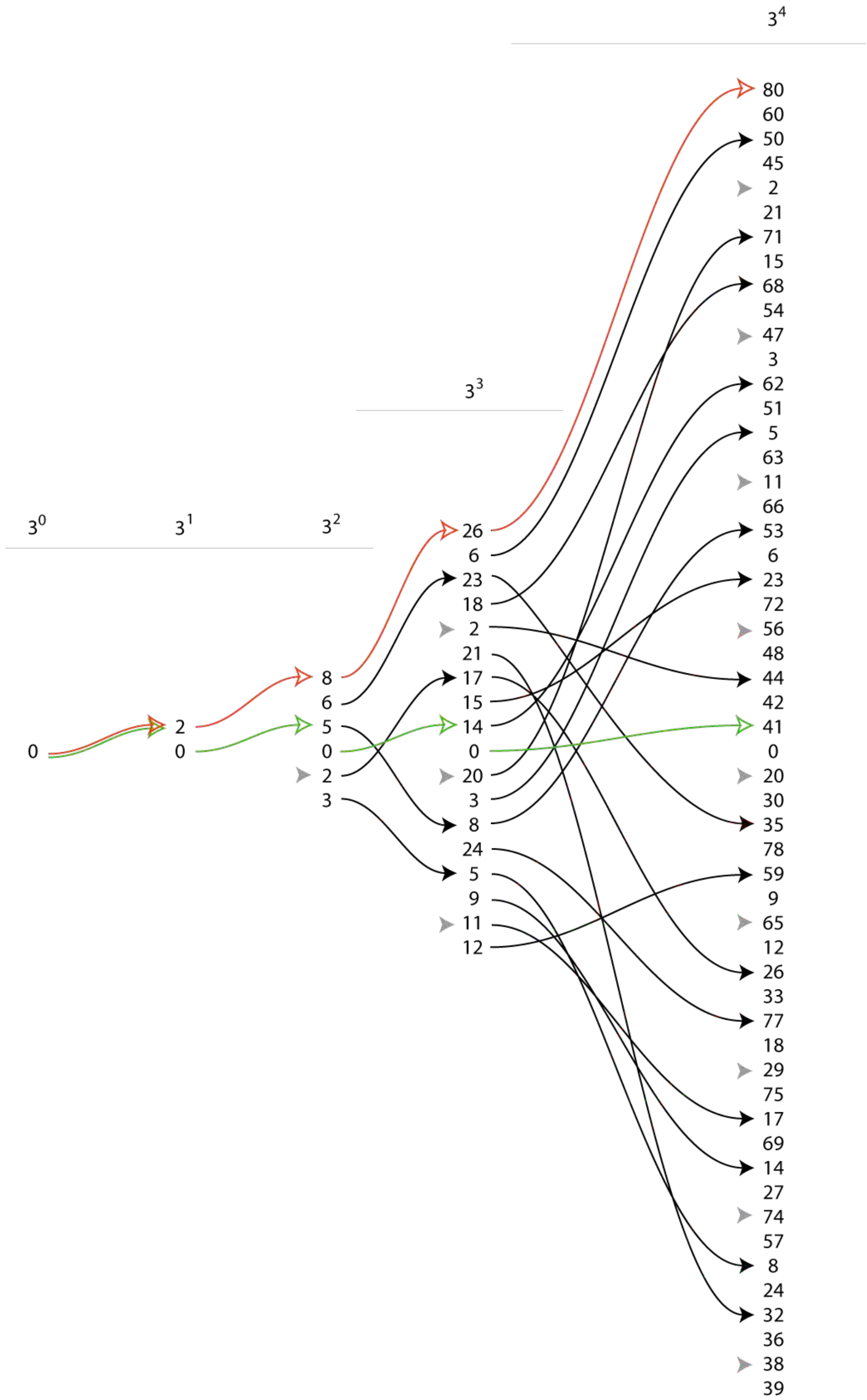

Fig. 11 – Summary of the fisrt four iterations. For each the possible odd index for the evenness decomposition is listed down (starting from 3^p-1), in their order of appearance. The arrows indicate how each subsetting series is first iterated into in the next decomposition (corresponding to an iterated evenness d=1).

we thus obtain the series :

$a(3,i) = \{\underline{26,6,23,18,2,21,17,15,14,0,20,3,8,24,5,9,11,12,}\ ...\}$, of period $2*3^2$,

$\varphi_3(i) = \{\underline{0,2,12,8,6,14,}\ ...\}$ (period $2*3^1$),

$a(4,i) = \{80, 60, 50, 45, 2, 21, 71, 15, 68, 80, 60, 50, 45, 2, 21, 71, 15, 68, 54, 47, 3, 62, 51, 5, 63, 11, 66, 53, 6, 23, 72, 56, 48, 42, 41, 0, ...\}$

of period $2*3^3$, starts to be too long to be written (it can be read more easily in the figure),

$\varphi_4(i) = \{\underline{0,2,30,8,24,50,36,20,12,26,6,14,18,38,48,44,42,32,}\ ...\}$ (period $2*3^2$).

We again find that the *a* series are self-symmetric series with all the possible values except the one corresponding to multiples of 3.

From this digram we can deduce as above the generators for the frist trhee (or first four) iterations. This gives the table below, for the first three iterations generators, limiting oursleve to the cube $i, j, k \in \{1, ..., 9\}$. It is similar to the previous table, in particular the general increase of the generators, with similar limits. Of course all the previous generators $g_2(i, j)$ are also present in this table, with their respective next iterated evenness. A difference is that now some generators are not iterated into other generators. The first exemple is $27 = g_3(1,2,1)$ which first iteration, 41, is not a third iteration generator.

k= 1

i \ j	1	2	3	4	5	6	7	8	9
1	15	27	19	67	163	99	995	1763	1251
2	9	33	81	49	497	881	625	2161	5233
3	61	109	77	269	653	397	3981	7053	5005
4	37	133	325	197	1989	3525	2501	8645	20933
5	245	437	309	1077	2613	1589	15925	28213	20021
6	149	533	1301	789	7957	14101	10005	34581	83733
7	981	1749	1237	4309	10453	6357	63701	112853	80085
8	597	2133	5205	3157	31829	56405	40021	138325	334933
9	3925	6997	4949	17237	41813	25429	254805	451413	320341

k= 2

i \ j	1	2	3	4	5	6	7	8	9
1	7	43	115	3	291	355	483	2787	7395
2	57	1	145	177	241	1393	3697	113	9329
3	29	173	461	13	1165	1421	1933	11149	29581
4	229	5	581	709	965	5573	14789	453	37317
5	117	693	1845	53	4661	5685	7733	44597	118325
6	917	21	2325	2837	3861	22293	59157	1813	149269
7	469	2773	7381	213	18645	22741	30933	178389	473301
8	3669	85	9301	11349	15445	89173	236629	7253	597077
9	1877	11093	29525	853	74581	90965	123733	713557	1893205

k= 3

i \ j	1	2	3	4	5	6	7	8	9
1	55	11	51	387	547	867	3555	739	3299
2	25	193	273	433	1777	369	1649	12401	17521
3	221	45	205	1549	2189	3469	14221	2957	13197
4	101	773	1093	1733	7109	1477	6597	49605	70085
5	885	181	821	6197	8757	13877	56885	11829	52789
6	405	3093	4373	6933	28437	5909	26389	198421	280341
7	3541	725	3285	24789	35029	55509	227541	47317	211157
8	1621	12373	17493	27733	113749	23637	105557	793685	1121365
9	14165	2901	13141	99157	140117	222037	910165	189269	844629

k= 4

i \ j	1	2	3	4	5	6	7	8	9
1	87	203	179	131	35	1891	5603	13027	11491
2	89	65	17	945	2801	6513	5745	4209	1137
3	349	813	717	525	141	7565	22413	52109	45965
4	357	261	69	3781	11205	26053	22981	16837	4549
5	1397	3253	2869	2101	565	30261	89653	208437	183861
6	1429	1045	277	15125	44821	104213	91925	67349	18197
7	5589	13013	11477	8405	2261	121045	358613	833749	735445
8	5717	4181	1109	60501	179285	416853	367701	269397	72789
9	22357	52053	45909	33621	9045	484181	1434453	3334997	2941781

k= 5

i \ j	1	2	3	4	5	6	7	8	9
1	23	331	435	643	3107	8035	1507	21219	27875
2	217	321	1553	4017	753	10609	13937	20593	99441
3	93	1325	1741	2573	12429	32141	6029	84877	111501
4	869	1285	6213	16069	3013	42437	55749	82373	397765
5	373	5301	6965	10293	49717	128565	24117	339509	446005
6	3477	5141	24853	64277	12053	169749	222997	329493	1591061
7	1493	21205	27861	41173	198869	514261	96469	1358037	1784021
8	13909	20565	99413	257109	48213	678997	891989	1317973	6364245
9	5973	84821	111445	164693	795477	2057045	385877	5432149	7136085

k= 6

i \ j	1	2	3	4	5	6	7	8	9
1	407	587	947	3715	1059	3939	26083	37603	60643
2	473	1857	529	1969	13041	18801	30321	118897	33905
3	1629	2349	3789	14861	4237	15757	104333	150413	242573
4	1893	7429	2117	7877	52165	75205	121285	475589	135621
5	6517	9397	15157	59445	16949	63029	417333	601653	970293
6	7573	29717	8469	31509	208661	300821	485141	1902357	542485
7	26069	37589	60629	237781	67797	252117	1669333	2406613	3881173
8	30293	118869	33877	126037	834645	1203285	1940565	7609429	2169941
9	104277	150357	242517	951125	271189	1008469	6677333	9626453	#

k= 7

i \ j	1	2	3	4	5	6	7	8	9
1	663	1099	4019	1667	5155	28515	42467	70371	257251
2	2009	833	2577	14257	21233	35185	128625	53361	164977
3	2653	4397	16077	6669	20621	114061	169869	281485	1029005
4	8037	3333	10309	57029	84933	140741	514501	213445	659909
5	10613	17589	64309	26677	82485	456245	679477	1125941	4116021
6	32149	13333	41237	228117	339733	562965	2058005	853781	2639637
7	42453	70357	257237	106709	329941	1824981	2717909	4503765	#
8	128597	53333	164949	912469	1358933	2251861	8232021	3415125	#
9	169813	281429	1028949	426837	1319765	7299925	#	#	#

k= 8

i \ j	1	2	3	4	5	6	7	8	9
1	1175	75	1971	5763	29731	44899	75235	4835	126179
2	985	2881	14865	22449	37617	2417	63089	184433	951409
3	4701	301	7885	23053	118925	179597	300941	19341	504717
4	3941	11525	59461	89797	150469	9669	252357	737733	3805637
5	18805	1205	31541	92213	475701	718389	1203765	77365	2018869
6	15765	46101	237845	359189	601877	38677	1009429	2950933	#
7	75221	4821	126165	368853	1902805	2873557	4815061	309461	8075477
8	63061	184405	951381	1436757	2407509	154709	4037717	#	#
9	300885	19285	504661	1475413	7611221	#	#	1237845	#

k= 9

i \ j	1	2	3	4	5	6	7	8	9
	2199	6219	14259	13955	13347	12131	140771	398051	912611
	7129	6977	6673	6065	70385	199025	456305	446577	427121
	8797	24877	57037	55821	53389	48525	563085	1592205	3650445
	28517	27909	26693	24261	281541	796101	1825221	1786309	1708485
	35189	99509	228149	223285	213557	194101	2252341	6368821	#
	114069	111637	106773	97045	1126165	3184405	7300885	7145237	6833941
	140757	398037	912597	893141	854229	776405	9009365	#	#
	456277	446549	427093	388181	4504661	#	#	#	#
	563029	1592149	3650389	3572565	3416917	3105621	#	#	#

Fig.12–Table of the generators $g_3(i, j,k)$ of the series of numbers with first history $\{i, j,k\}$, with $i, j,k \in \{1, 9\}$.

<u>8 - Simplification</u>

These graphs look complex. However we can simplify them by writing down the two relationships that links one iteration to the next, and the generators within one period.
When the iteration of one odd subset generators $a(n, j)$ to the next $a(n+1, l)$ is drawn not in their periodic order but in simple numerical order (forgetting the phases φ), it simplifies to a very simple function, here for the second and third iterations:

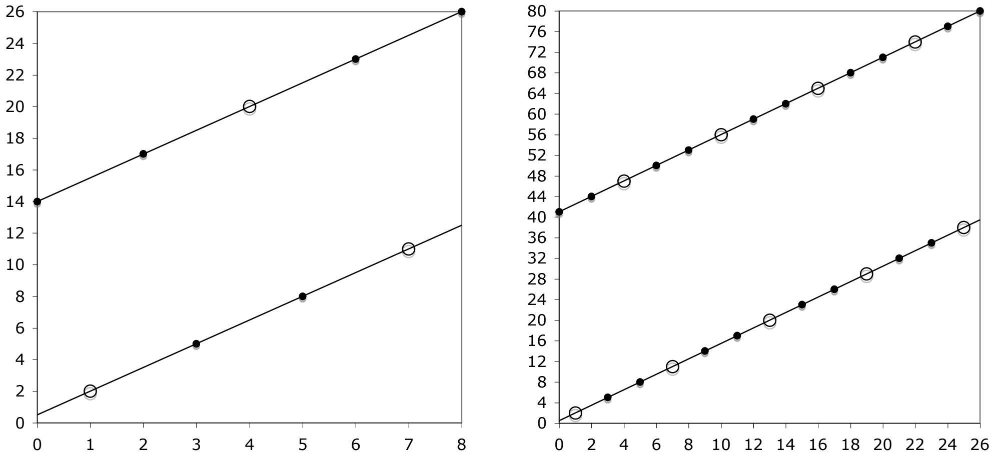

Fig. 13 – iteration of one odd index of the decomposition of nth iteration, to the odd number of decomposition of the next iteration n+1, with iterated evenness 1, for the case from the second to third iteration, (left) and third to fourth iteration (right).

Similarly, the third and forth series can be simplified when the new term $a(n, j+1)$ is plotted as a function of $a(n, j)$:

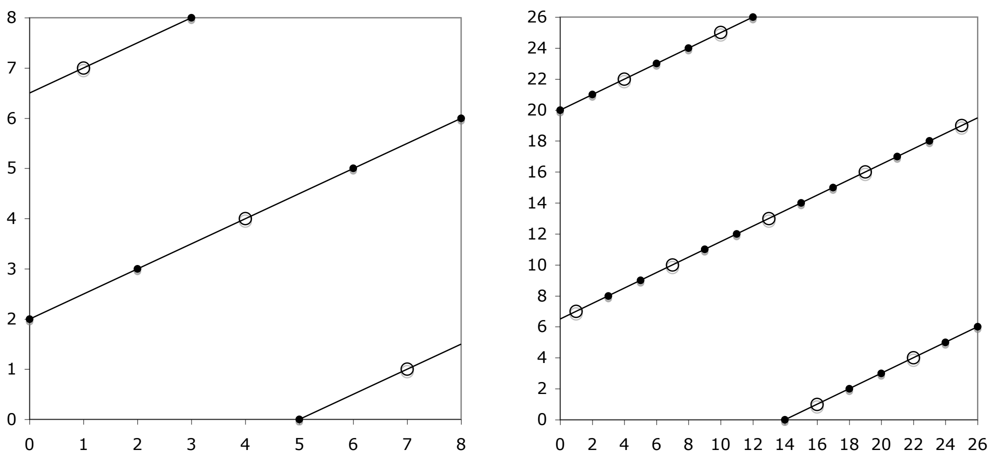

Fig. 14 – Function of iteration of one odd number to the next in the decompositions of the result of a given iteration, for the case of the second iteration (left) and he third iteration (right).

In fact these functions can be even more simplified by noticing that they correspond to single linear functions folded modularly by the maximum period 3^n (here 27 in the third iteration):

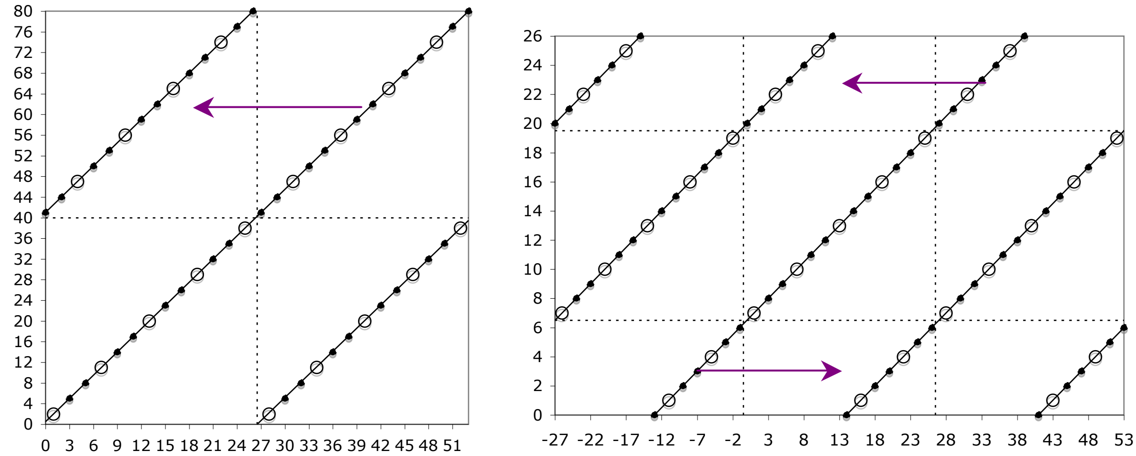

fig. 15 – "Unfolded" drawing of the previous functions. On the right, the function from an odd decomposition index of the third iteration to the fourth iteration (with iterated evenness 1), same as fig. 13 left, but doubled to show the unique function folded back through the period 3^3 (arrow). On the left, the iteration function with increasing eveness of the decomposition of the third iteration, tripled, to show the double folding of a unique function with the period 3^3 (arrows) as in fig. 14 left and fig. 3.

The distinction between odd or even absica come from the fat that the linear function is true only for period 2 absissa, which correspond to a give parity in one period, and to the complementary one in the surrounding (odd) periods.

9 - Generalisation

The previous relationships can be computed directly and explicitly, thanks to T and the odd/even decomposition. The first relationship is between one generator of the nth iterate $a(n,j)$ and the first one in the iteration decomposition, so with $d=1$, $a(n+1,l)$. In order to avoid the complex previous notation with the phases, which runs from the maximum a value (useless here), we will just note them a_n and $a_{n+1,1}$. So let us start from the result series $S(1,2)//S(a_n,3^n)$, as we found from the first two (four) iterations.
Applying the first part of the transformation gives:
$T(S(1,2)//S(a_n,3^n)) = T(S(1+2*a_n,2*3^n)) = S(4+2*3*a_n,2*3^{n+1}) = 2*S(2+3*a_n,3^{n+1})$
As the new period (after division by 2) is odd, it contains both odd and even results, and we will have to decompose it to infinity (*property 9*).
The first step depends on the parity of $2+3*a_n$, so in fact on the parity of a_n .

- If a_n is odd the first odd subset is given by (*property 7*) :
$S(1,2)//S((-1+2+3*a_n)/2,3^{n+1})$, or in other words,
$a_{n+1,1} = (1+3*a_n)/2$
- If a_n is even the first odd subset is given by (*property 7*) :
$S(1,2)//S((-1+2+3*a_n+3^{n+1})/2,3^{n+1})$, or in other words,
$a_{n+1,1} = (1+3*a_n+3^{n+1})/2$

We can summarize this by :

Property 18

$$\left\langle \begin{array}{ll} e(a_n)=0: & a_{n+1,1} = (1+3*a_n)/2 \\ e(a_n)\geq 1: & a_{n+1,1} = (1+3*a_n+3^{n+1})/2 \end{array} \right.$$

which is the same function $g(a) = (1+3*a)/2$ shifted by a period $q=3^n$,
$g(a+q) = (1+3*a+3^{n+1})/2$, depending on the parity.

These are the two lines that are drawn above in *fig. 13*. From the translation we know that the two lines are exactly one above the other. This is interesting as even if the direct iteration of this function is complex (depending on the parity), the reverse function is thus very simple. The maximum value $a_n = -1+3^n$ gives the next maximum result $a_{n+1,1} = (-2+2*3^{n+1})/2 = -1+3^{n+1}$
The minimum value is obtained for the minimum odd number $a_n = 1$ which gives $a_{n+1,1} = 2$, but this corresponds to a multiple of 3, so the first really possible a value is obtained for $a_n = 3$ which gives $a_{n+1,1} = 5$.

Another detail is that this function works also perfectly for the forbidden original generators multiple of 3, giving the "shadow" images (grey arrowhead in *fig. 9,11* and empty circles in *fig. 13-14*).

Similarly, the iteration of the generators from iterate $a(n,j)$ to $a(n,j+1)$, wherever we start the origin of the phase (either locally as above or globally from the maximum value as in *fig. 9,11*), can be written explicitly from the odd/even decomposition. It is the decomposition written previously in *property 8*, here for the particular case of $q=3^n$:

Property 19

$$
\begin{cases}
e(3^n-1)=1 & \begin{cases} e(a_{n,j})=0: & a_{n,j+1}=a_{n,j}/2+(3^n-1)/4 \\ e(a_{n,j})\geq 1 & \begin{cases} a_{n,j}<(3^n-1)/2: & a_{n,j+1}=(a_{n,j}+1)/2+3(3^n-1)/4 \\ a_{n,j}\geq(3^n-1)/2: & a_{n,j+1}=(a_{n,j}-1)/2-(3^n-1)/4 \end{cases} \end{cases} \\
e(3^n-1)\geq 2 & \begin{cases} e(a_{n,j})=0 & \begin{cases} a_{n,j}<(3^n-1)/2: & a_{n,j+1}=(a_{n,j}+1)/2+3(3^n-1)/4 \\ a_{n,j}\geq(3^n-1)/2: & a_{n,j+1}=(a_{n,j}-1)/2-(3^n-1)/4 \end{cases} \\ e(a_{n,j})\geq 1: & a_{n,j+1}=a_{n,j}/2+(3^n-1)/4 \end{cases}
\end{cases}
$$

so that it is always the same linear function $f_q(a)=a/2+(q-1)/4$, , in the special case of $q=3^n$, with only a possible shift in the abscissa by *q*, $f_q(a\pm q)$, depending on the parities.

The evenness of 3^n-1 changes, as we can see in *fig. 14* for *n* = 3 and 4. If we plot it we find a period two for $e(3^n-1)=1$ and $e(3^n-1)\geq 2$. More precisely it seems to give a fractal structure similar to $e(n)$, with the difference that we have no evenness 2 (and evenness 3 has a period 4 instead of 8, and corresponding following shift). Appendix ***B*** demonstrates this up to evenness 4. The fact that we find he same type of fractal structure is intresting as now we are not making a linear sampling as in the subseting operation, but a exponential sampling, increasing each time by a factor 3.

The two shifts correspond to the three lines of the previous graphs. Again the three lines are shifted just one above the others, without overlapping, making the inverse function a simple one, as drawn in *fig. 3* and *15*. These three lines can be seen also when the *a* series for all the possible $S(p,q)$ for a given *q* are plotted, as in *fig. 4, 6*. We can see that one parity is just reduced by two around the central symmetry line, while the other parity is shifted on the other side of the line. In this way it always creates a good “mixing” of the lines, that allows to create a self holding interlaced ribbon, as in *fig. 4, 6*. This is this mixing that produces the appearant “stochasticity” of the iterations, as described in Lagarias (1985).

The overall shape of these functions of *property 18* and *19*, when connected point to point, present a jigsaw pattern with slopes larger than 1, explaining the origin of the stochasticity as a classic iterated map generating chaos. On top of that, to induce more appearent stochasiticy, is the fact that these functions evolve for each iteration.

10 - Generators and periods

Similarly as for the previous two iterations, knowing the $a_{n,j}$ allows to compute explicitly the series of numbers that share the same beginning of history. Each iteration corresponds to a new subsetting.
We can thus find the general formula by iteration. Let us assume that the result after *m* iterations of the numbers $G(h_m)$ having a particular history h_m is
$C^m(G(h_m)) = S(1,2)//S(a_m, 3^m)$
as we have obtained with the first four iterations. Then, at the next iteration, we obtain a series
$T(C^m(G(h_m))) = T(S(1,2))//S(a_m, 3^m) = 2*S(2,3)//S(a_m, 3^m) = 2*S(2+3a_m, 3^{m+1})$
which is twice a series with an odd period, 3^{n+1}, so that it contains (*property 9*) all the evenness.
If we restrict ourselves to a given one, i_{m+1}, leading to the last term of the new history h_{m+1}, it correspond to the subset $2^{i_{m+1}} * S(1,2)//S(a_{m+1}, 3^{m+1})$ in the decomposition and finally a result for the iteration $C^{m+1}(G(h_{m+1})) = S(1,2)//S(a_{m+1}, 3^{m+1})$.
Following *property 10*, this subset of given evenness also correspond to a subset of the original series by the relation $2^{i_{m+1}} * S(1,2)//S(a_{m+1}, 3^{m+1}) = 2*S(2+3a_m, 3^{m+1})//S(x_{m+1}, 2^{i_{m+1}})$
Or in other words $x_{m+1} = \left[-2 - 3*a_m + 2^{i_{m+1}-1}(1+2a_{m+1})\right]/3^{m+1}$.
Now, as $C(G)//S = C(G//S)$ (*property 2*), we can deduce that
$G(h_{m+1}) = G(h_m)//S(x_{m+1}, 2^{i_{m+1}})$

From this we can deduce the general form for the numbers sharing the same beginning of history. For instance, all the numbers that have $h_3 = \{i, j, l\}$ as first history are the series :
$G(i,j,l) = S(1,2)//S(x_1(i), 2^i)//S(x_2(i,j), 2^j)//S(x_3(i,j,l), 2^l)$
so with a period
$q(i,j,l) = 2^{1+i+j+l}$
and a generator
$g_3(i,j,l) = 1 + 2x_1(i) + 2^{1+i}x_2(i,j) + 2^{1+i+j}x_3(i,j,l)$
with
$x_1(i) = \left(-2 + 2^{i-1}(1+2a_1(i))\right)/3^1$,
$x_2(i,j) = \left[-2 - 3*a_1(i) + 2^{j-1}(1+2a_2(j+\varphi_1(i)))\right]/3^2$,
$x_3(i,j,l) = \left[-2 - 3*a_2(j+\varphi_1(i)) + 2^{l-1}(1+2a_3(l+\varphi_2(j+\varphi_1(i))))\right]/3^3$
In fact the formula at a given order depends explicitly on the subsetting generator *a* at this order and at the previous order, as well as the value of the last history. The dependence on the previous history is there just to be able to find the right generators *a* by expressing the right phase. If we resume this phase dependence with the history up to this order we can write it in a more compact form :
$x_1(h_1) = \left(-2 + 2^{i-1}(1+2a_1(h_1))\right)/3^1$,
$x_2(h_2) = \left[-2 - 3*a_1(h_1) + 2^{j-1}(1+2a_2(h_2))\right]/3^2$,
$x_3(h_3) = \left[-2 - 3*a_2(h_2) + 2^{l-1}(1+2a_3(h_3))\right]/3^3$
and we could even remove the reference to the history, being understood that it is necessary to find the right *a*.

As it is the same operation at each step, the general result can be easily deduced. To write it in the simplest way, we can first define, for an history $h_p = \{i_1, i_2, ..., i_m, ..., i_{p-1}, i_p\}$ all the partial sum: $\sigma_m = 1 + i_1 + i_2 + ... + i_m$

Then we can write more simply $q_p(h_p) = 2^{\sigma_p}$ and

$g_p(h_p) = 1 + 2x_1 + 2^{\sigma_1} x_2 + ... + 2^{\sigma_{m-1}} x_m + ... + 2^{\sigma_{p-1}} x_p$

We can also define an $a_0 = 0$, so that the first expression for x_1 is the same as for x_m.

Similarly, we can define an $i_0 = 1$ and have a compact expression for $\sigma_p = \sum_{m=0}^{p} i_m$, and an $x_0 = 1$ (and an $\sigma_0 = 0$) so that $g_p(h_p) = 2^0 x_0 + 2^{\sigma_0} x_1 + 2^{\sigma_1} x_2 + ... + 2^{\sigma_{p-1}} x_p = \sum_{q=0}^{p} 2^{\sigma_{q-1}} x_q$.

With these notations, the general result reads:

Property 20 (general formula for numbers of identical beginning of history)

The numbers having a first history $h_p = \{i_1, i_2, ..., i_m, ..., i_{p-1}, i_p\}$, are the series

$G(h_p) = S(1,2)//S(x_1, 2^{i_1})//S(x_2, 2^{i_2})//...//S(x_m, 2^{i_m})//...//S(x_{p-1}, 2^{i_{p-1}})//S(x_p, 2^{i_p})$

with, for $1 \le m \le p$,

$x_m(h_m) = \left[-2 - 3 * a_{m-1}(h_{m-1}) + 2^{i_m - 1}(1 + 2a_m(h_m))\right]/3^m$

where the $a_m(h_m)$ can be obtained with *property 18* and *19*, and $a_0 = 0$.

This is thus a series of period

$q_p(h_p) = 2^{\sigma_p}$ with $\sigma_p = \sum_{m=0}^{p} i_m$ and $i_0 = 1$

and generator $g_p(h_p) = \sum_{q=0}^{p} 2^{\sigma_{q-1}} x_q$ with $x_0 = 1$ (and $\sigma_{-1} = 0$).

The result of the successive iterations are

$G(h_p) = S(1,2)//S(x_1, 2^{i_1})//S(x_2, 2^{i_2})//...//S(x_m, 2^{i_m})//...//S(x_{p-1}, 2^{i_{p-1}})//S(x_p, 2^{i_p})$

$C(G(h_p)) = S(1,2)//S(a_1, 3^1)//S(x_2, 2^{i_{21}})//...//S(x_m, 2^{i_m})//...//S(x_{p-1}, 2^{i_{p-1}})//S(x_p, 2^{i_p})$

$\vdots$

$C^m(G(h_p)) = S(1,2)//S(a_m, 3^m)//...//S(x_{p-1}, 2^{i_{p-1}})//S(x_p, 2^{i_p})$

$\vdots$

$C^{p-1}(G(h_p)) = S(1,2)//S(a_{p-1}, 3^{p-1})//S(x_p, 2^{i_p})$

$C^p(G(h_p)) = S(1,2)//S(a_p, 3^p)$

We can develop the expression for the generator, and find that the successive a_m vanish except the last one. For instance,

$g_4(i,j,l,m) = -(1/3)*(1 + 2^i/3^1 + 2^{i+j}/3^2 + 2^{i+j+l}/3^3) + 2^{i+j+l+m}/3^4 + 2^{1+i+j+l+m} a_4/3^4$

In general, the generator of history h_p can thus be expressed as :

$$g_p(h_p) = -\frac{1}{3}\left(\sum_{m=0}^{p-1}\left(\frac{2^{\sigma_m-1}}{3^m}\right)\right) + \frac{2^{\sigma_p-1}}{3^p} + \frac{2^{\sigma_p}a_p}{3^p}$$

This formula is not very convenient because, independently of dependency on the value of a_p, it is the sum of negative and positive terms, both increasing with the iteration number, so its value is not a priori clear.

Property 20 performs the idea of Terras (1976) that knowing the beginning of history is enough information to define a set of numbers sharing this beginning of history. In this case it gives an explicit (even though iterative, using *property 18* and *19*) way to construct theses series, the iteration being only on the fractal structure parameters (the 'a's). The general formula recover the result on the periodicity of such sets obtained by Lagarias (1985), and extends it as giving also the first number of these series.

11 - Two trajectories

Before going further, let us come back to two particular trajectories in the iterations, drawn in red and green in *fig. 11*. As we saw in the iteration function (*property 18*), after p iterations the maximum a value $a_p = -1+3^p$ gives directly (corresponding to $i_{p+1} = 1$) the next maximum $a_{p+1} = -1+3^{p+1}$. This corresponds to the red trajectory in *fig. 11*. This thus corresponds to a history $h_p = \{1,1,1,...,1\} = \{(1,)^p\}$, or $i_n = 1$ for all $n \le p$.

If we combine $a_p = -1+3^p$ and $i_p = 1$ in the expression of *property 20* for x_p :

$x_p(h_p) = \left[-2-3*a_{p-1}(h_{p-1}) + 2^{i_p-1}(1+2a_p(h_p))\right]/3^p$, this gives along this trajectory

$x_p(\{1,\}^p) = \left[-2-3*(-1+3^{p-1}) + 2^0*(1+2*(-1+3^p))\right]/3^p = 1$

So this trajectory correspond to a constant $x_p = 1$, and a constantly increasing generator, as we find

$g_n(\{1,\}^n) = 1+2+2^2+...+2^n = 2^{n+1}-1$.

We also know the period, $q_n(\{1,\}^n) = 2^{n+1}$, so it corresponds to the series $G(\{1,\}^n) = S(2^{n+1}-1, 2^{n+1})$. We can summarize that by:

Property 21

After p iterations of C, the maximum possible value for a is $a_p = -1+3^p$, and this maximum value gives directly (corresponding to $i_{p+1} = 1$) the next maximum $a_{p+1} = -1+3^{p+1}$. This correspond to a history $h_p = \{(1,)^p\}$, with a constant $x_p = 1$, a generator $g_n(\{1,\}^n) = 2^{n+1}-1$, and a period $q_n(\{1,\}^n) = 2^{n+1}$, in other words to the series $G(\{1,\}^n) = S(2^{n+1}-1, 2^{n+1})$.

Let us write the exact number of repetition n time of $d = 1$, exactly (so that the next iteration will not be 1), of a number p, as $r_1(p)$. In the same time $G(\{1,\}^n) = G(\{1,\}^n)//S(0,2) \cup G(\{1,\}^n)//S(1,2) = S(2^{n+1}-1, 2^{n+2}) \cup G(\{1,\}^{n+1})$. So the series for a given r_1 is $G(r_1 = n) = S(2^{n+1}-1, 2^{n+2})$. We can also notice that $1+G(r_1 = n) = 2^{n+1}*S(1,2)$ so these are all the numbers of evenness $n+1$. This gives the otherwise surprising property, as it link a number of iteration with the value of evenness of a number:

Property 22

$r_1(p) = e(p+1) - 1$
In other words, the exact number of times a number p will be iterated successively with $d=1$ is equal to the evenness of $p+1$, lowered by 1.

For instance for $p = 7$, we have $e(8) = 3$ so it is indeed iterated twice with $d=1$, before having a different value of d. Another example, taken form the iterations of 27, is 319, as we have $319+1 = 5*2^6$ i.e. $e(320) = 6$, so it will iterate 5 times with $d=1$ before having another d value.

The other particular trajectory corresponds to the iteration with $a_n = 0$. We can notice that the decomposition of the iteration starting from a null subsetting generator, $a_n = 0$, gives also $a_{n+1,2} = 0$ for $d = 2$.
In other words $S(1,2)//S(0,3^n)$ will give in the next decomposition $2^2 * S(1,2)//S(0,3^{n+1})$:

$T\left(S(1,2)//S(0,3^n)\right) = S(4, 2*3)//S(0,3^n) = 2*S(2,3^{n+1})$
$S(2,3^{n+1}) = S(2,3^{n+1})//S(0,2) \cup S(2,3^{n+1})//S(1,2)$ where
$S(2,3^{n+1})//S(0,2) = S(2,2*3^{n+1}) = 2*S(1,3^{n+1})$ is even and
$S(1,3^{n+1}) = S(1,3^{n+1})//S(0,2) \cup S(1,3^{n+1})//S(1,2)$ gives
$S(1,3^{n+1})//S(0,2) = S(1,2)//S(0,3^{n+1})$ odd (a possible commutation of the periods when the subsetting generator is null). We could add that the decomposing generator for $d=1$ is $(1+3^{n+1})/2$: $S(2,3^{n+1})//S(1,2) = S(2+3^{n+1}, 2*3^{n+1}) = S(1,2)//S((1+3^{n+1})/2, 3^{n+1})$

This corresponds to the green trajectory in *fig. 11*. We can also remember that a succession of $d=2$ indefinitely correspond to the history of 1. We can check that by looking at the generators of this trajectory: we now have $a_p = 0$ and $i_p = 2$ in the expression of *property 20* for x_p :
$x_p(h_p) = \left[-2 - 3*a_{p-1}(h_{p-1}) + 2^{i_p - 1}(1+2a_p(h_p))\right]/3^p$, this gives along this trajectory $x_p(\{2,\}^n) = \left[-2 - 3*0 + 2^1*(1+2*0)\right]/3^p = 0$. So we have again a constant x_p, but now equal to 0. The generator thus remains constant and equal to 1 : $g_n(\{2,\}^n) = 1$
The period is increasing as twice the one from the previous trajectory, i.e. $q_n(\{2,\}^n) = 2^{1+i_1+\ldots+i_n} = 2^{2n+1}$. We can summarize this into :

Property 23
Along the iterations of C, the minimum possible value for a is $a_p = 0$, and each minimum value gives at the second step in decomposition (corresponding to $i_{p+1} = 2$) the next minimum $a_{p+1} = 0$. This correspond to a history $h_p = \{(2,)^p\}$, a constant $x_p = 0$, thus a constant generator $g_n(\{2,\}^n) = 1$, and a period $q_n(\{2,\}^n) = 2^{2n+1}$, in other words to the series $G(\{2,\}^n) = S(1, 2^{2n+1})$.

12 - Generator increase

The figure 10 allows to read the successive a_p after *p* iterations with a given history $h_p = \{i_0, i_1, i_2, \dots i_p\}$. From that we can also compute the successive x_p (with *property 20*), and thus generators g_p and period q_p of the series $G(h_p) = S(g_p, q_p)$ sharing the same beginning of history. We can note the successive generators $F(h_p) = \{g_0, g_1, g_2, \dots g_p\}$.
With the previous formula on the generator of *property 20*, for a given history h_p :
$g_p(h_p) = \sum_{q=0}^{p} 2^{\sigma_{q-1}} x_q$, with $\sigma_p = \sum_{m=0}^{p} i_m$.
We can see that globally the generator will increase considerably, like the *power of 2 of the sum of the history,* at the condition that x_p *is not zero.* If we look at our previous example of two particular trajectories, for the first one we have a constant $i_p = 1$ and $x_p = 1$, and for the second one also a constant $i_p = 2$ and a more interesting constant $x_p = 0$.

So the interesting point is to see when x_p can be equal to 0. Following *property 20* we have $x_p(h_p) = \left[-2 - 3 * a_{p-1}(h_{p-1}) + 2^{i_p - 1}(1 + 2a_p(h_p))\right] / 3^p$.
Generally, having $x_p = 0$ leads to the condition
$2^{i_p - 1}(1 + 2 * a_p(h_p)) = 2 + 3 * a_{p-1}(h_{p-1})$
An interesting particularity of this formula is that it does not depend on the number of iterations *p*. This condition thus remains the same, and with the successive iterations the only thing that increase is the maximum possible value of a_n : ($-1 + 3^n$).

This condition can be rewritten as
$(1 + 2 * a_p(h_p)) = (1 + 3 * (1 + 2a_{p-1}(h_{p-1}))) / 2^{i_p}$
so a_p and a_{p-1} are the odd index of the numbers having an iterated eveness $d = i_p$ through the Collatz iteration. We have seen above, when studying the first iteration, that for a given iterated evenness $d = i$ such numbers are the series $G(\{i\}) = S(1,2) // S(x_{i-1}, 2^i)$,
with for $i = 2j + 1$ odd : $x_{2j+1} = 2 * \left(-1 + 2^{2j}\right) / 3$,
and for $i = 2j$ even : $x_{2j} = 2 * \left(-1 + 5 * 2^{2j-1}\right) / 3$
This gives for the odd indexes the subsetting series :
for $i = 0$: $S(1,2)$, for $i = 1$: $S(0,4)$,
for $i = 2$: $S(6,8)$, for $i = 3$: $S(2,16)$,
for $i = 4$: $S(26,32)$, for $i = 5$: $S(10,64)$ … as can be seen in *figure 16*.

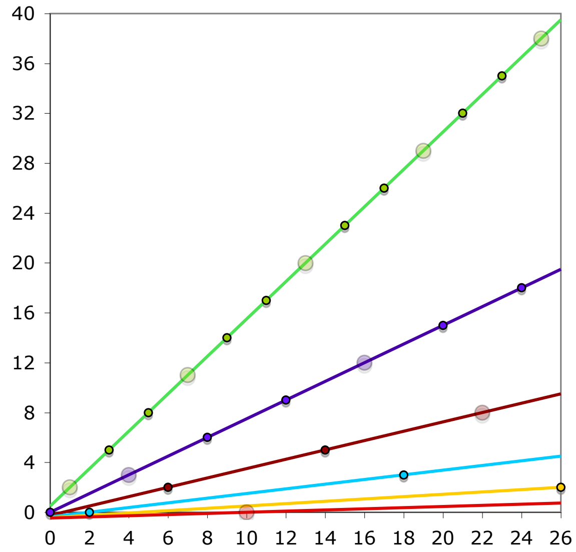

Fig. 16 – link between one odd index and the one at the next iteration so that x=0, for increasing iterated evenness (green for d=1, violet d=2, bordeaux d=3, blue d=4, yellow d=5, red d=6). This figure is in fact the perect image of the first iteration of C (once back to the odd numbers n=1+2a).

If we recall that the result after p iteration is (property 20):
$C^p(G(h_p)) = S(1,2)//S(a_p,3^p) = S(1+2*a_p,2*3^p)$,
so the generator after p iteration is $C^p(g_p) = (1+2*a_p)$,
the above formula can be rewritten:
$C^p(g_p) = T(C^{p-1}(g_{p-1}))/2^{i_p}$ or $C^p(g_p) = C(C^{p-1}(g_{p-1})) = C^p(g_{p-1})$
so $g_p = g_{p-1}$ which is the definition of $x_p = 0$.
in other words the new number is the direct image through C of the previous one. For each value of i_p it gives a simple straight line as visible in fig. 16.

Any odd index a corresponds to a number $n = 1+2*a$ which can be iterated, and the corresponding iteration eveness is the one such that $x = 0$. In other words, for any odd index a there is a particular iterated evenness such that $x = 0$. We can combine this knowledge with the previous drawing of the fisrt iterations, drawing not only to which iterated evennes a is iterated with $d = 1$, but also to which odd index it is going with the correct iterated eveness so that $x = 0$:

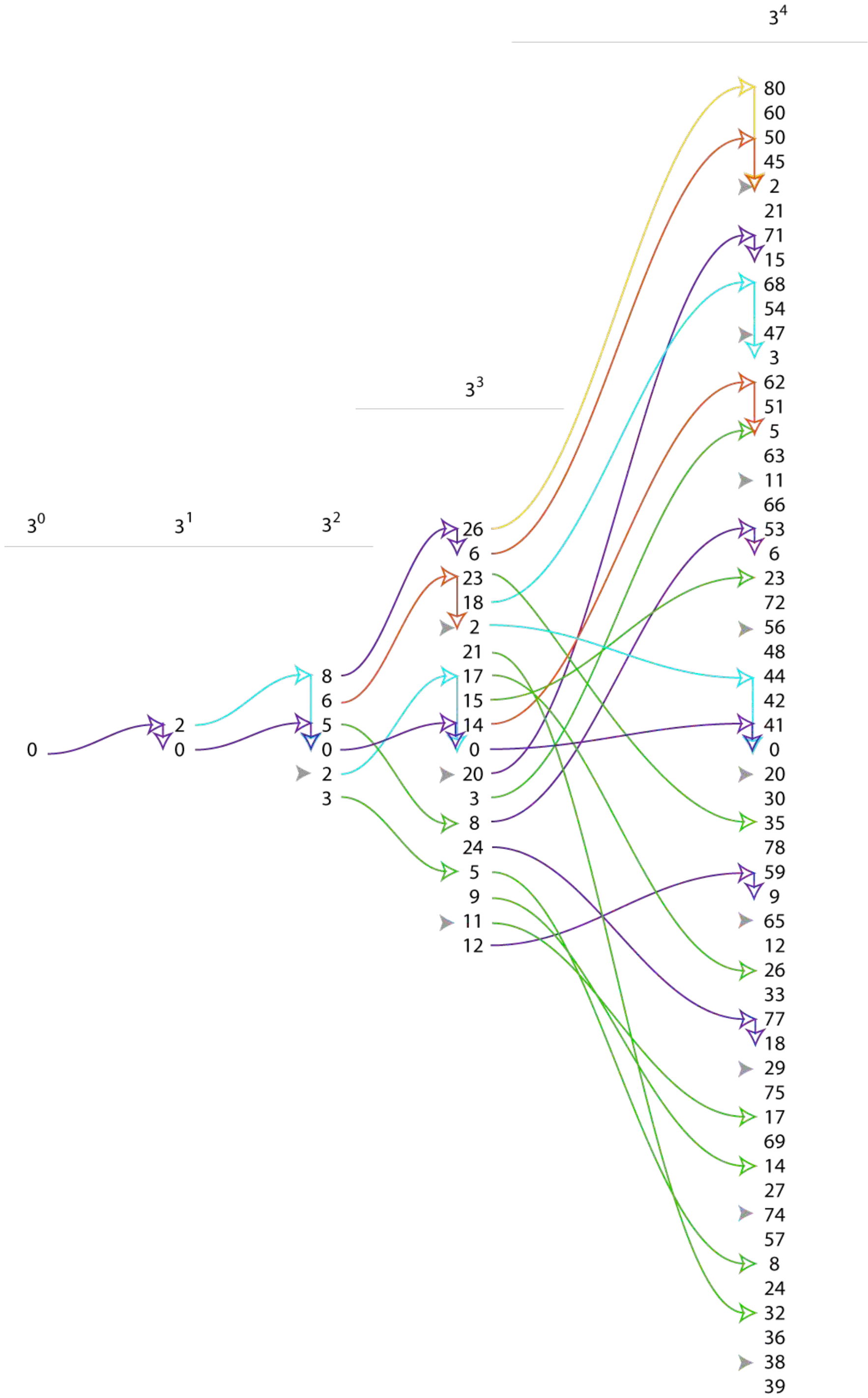

Fig. 17 – The diagram for the first four iterations, on which not only the periodic series of "a" for each number of iteration has been written (vertical columns), and their iteration into the next series with iterated evenness 1 indicated (curved horizontal arrows), but also to which evenness it would give exactly an x=0 (vertical arrows). Each corresponding iterated evenness has been coloured with the same colours as in fig. 16.

This property can happen any time in the iteration. However, there is a special case when the generator becomes equal to the original number : $g_m = n$. Then we know that the generator cannot increase anymore (it is the first number share the same beginning of history as n, it is at most equal to n), or in other words that forever after we have $x_p = 0$.

This condition can be also expressed as $C^m(g_m) = C^m(n)$. We know from property 20 that the series of similar history $G(h_m)$ is iterated into $C^m(G(h_m)) = S(1,2) // S(a_m, 3^m) = S(1+2*a_m, 2*3^m)$, so $C^m(g_m) = (1+2*a_m)$, and the condition can be rewritten as $C^m(n) = (1+2*a_m)$, or in other words it is the first term of this series (null index). We can translate this into the property :

Property 24

If a number n is such that after m iterations $C^m(n) = (1+2*a_m)$, this means that this number is equal to the generator of history h_m, $g_m(h_m) = n$. This relationship $C^p(n) = (1+2*a_p)$ will then hold on to infinity, with following x_p being null : $\forall p > m, \quad x_p = 0 \ \ and \ \ C^p(n) = (1+2*a_p)$

If m is the lowest number for which this property appears, then x_m is not null, as it is the last value added to the successive generators to reach the number n . Reciprocally $x_p = 0$ doesn't mean anything, as it can be just a pause in the generator increase. The real significant condition is either $g_m = n$ or equivalently $C^m(n) = (1+2*a_m)$.

We reached these results by looking at the numbers sharing the same history. In the same time, when one considers a given number, it can be iterated indefinitely, with a corresponding history. If now we look at this history and ask what are the numbers sharing the same history, there can be smaller numbers having the same beginning, but as the generator tends to increase quickly, each number will necessarily become a generator of his own history after a large enough number of iteration.

More precisely, for a given history, we can compute a generator, i.e. the first number that have this history for its beginning. Now on the next step, this number has a given iterated evenness i, and the other number of the series have a different one, or the same, periodically. The shift to another (higher) number corresponds to the fact of having an x different from zero for this new iterated evenness. The fact this generator can be iterated means that for his own history then all the x are necessarily null up to an infinite number of iterations. The other numbers that share the same history are getting further apart as the period is increasing, at least by a factor 2 at each iteration (for $i \geq 1$), even if x being zero.

This means that any number is necessarily after some time its own generator of his beginning of history. There can be smaller numbers sharing the same beginning of history, but as the period between such common numbers increases, it is soon impossible to have any smaller number far enough apart. Form property 20, writing the sum of history $\sigma_p = i_0 + i_1 + i_2 + ... + i_p$, the period is $q_p(h_p) = 2^{\sigma_p}$. As $1 \leq i_p$, $1+m \leq \sigma_m$, so $2^{m+1} \leq q_m$. Thus

Property 25 (generator limit)

Any number *n* is necessarily its own generator after at least $p_{\max}(n) = (\ln(n)/\ln(2) - 1)$ iterations. After these iterations, all the rest of history, up to infinity, corresponds to $x_m = 0$ and $C^m(n) = 1 + 2 * a_m$.

This is a good limit as it is reached for the numbers having a series of $d = 1$ at their beginning, which (property 21) are the $G(\{1,\}^n) = S(2^{n+1} - 1, 2^{n+1})$, corresponding after their n iterations to a period 2^{n+1} just above their own value $2^{n+1} - 1$, so just becoming generators. On the contrary some numbers starts with the largest possible *d*, which are the $(2^d - 1)/3$, and are already after one step their own generators. So the actual value is between 1 and this limit. Knowing the history of each number, we can refine the prediction at which it have to be a generator using the successive σ_m and also compare it to when it actually become a generator :

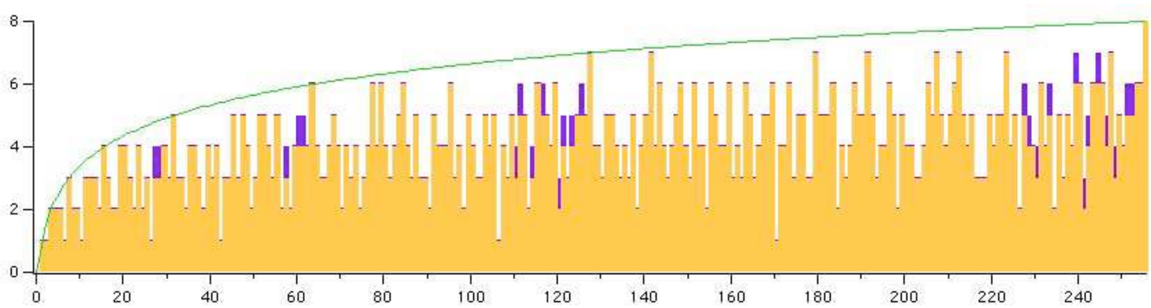

Fig. 18 – Number of iterations for which a number must become its own generator (in violet bars, for the first 256 odd numbers) because of the increase of total history, compared with the upper limit of property 24, and to the actual number of iterations at which a number becomes its own generator (when $C^m(n) = 1 + 2 * a_m$) in yellow. The upper limt delimits perfectly the curves and touch it for 2^m-1. The yellow and violet bars corresponds perfectly except at few places (slightly before 2^m-1) where a number becomes its own gerenator only one iteration before it is obliged to do so from the period increase.

For a more complete representation of a number's history, we should thus add the successive a_m values, as well as the successive x_m values :

Let us call $\overset{m}{D}(n) = \{a_0, a_1, a_2, \ldots a_m\}$, and $\overset{m}{Q}(n) = \{x_0, x_1, x_2, \ldots x_m\}$.

By convention, as said above, we can note $a_0 = 0$ and $x_0 = 1$ (we start from the odd numbers $S(1,2)//S(0,1)$), and $i_0 = 1$. A more complete description of the iteration of a number should thus be:

$$\frac{\overset{m}{H}}{\frac{Q}{\frac{D}{C}}}(n) = \left\{ \frac{1}{\frac{1}{\frac{0}{n}}}, \frac{i_1}{\frac{x_1}{\frac{a_1}{C(n)}}}, \ldots \frac{i_2}{\frac{x_m}{\frac{a_m}{C^m(n)}}} \right\}$$

This gives for instance

$$\frac{\overset{\infty}{H}}{\frac{Q}{\frac{D}{C}}}(7)=\left\{\frac{\mathit{1}}{\frac{\mathit{1}}{\frac{\mathit{0}}{7}}},\frac{1}{\frac{1}{\frac{2}{11}}},\frac{1}{\frac{1}{\binom{8}{17}}},\frac{2}{\frac{0}{\binom{6}{13}}},\frac{3}{\frac{0}{\binom{2}{5}}},\frac{4}{\frac{0}{\binom{0}{1}}},\frac{2}{\frac{0}{\underline{\binom{0}{1}}}},\ldots\right\}$$

where we underline between brackets when $C^m(n)=1+2*a_m$. Anoter exemple is

$$\frac{\overset{\infty}{H}}{\frac{Q}{\frac{D}{C}}}(27)=\left\{\frac{\mathit{1}}{\frac{\mathit{1}}{\frac{\mathit{0}}{27}}},\frac{1}{\frac{1}{\frac{2}{41}}},\frac{2}{\frac{2}{\frac{6}{31}}},\frac{1}{\frac{1}{\binom{23}{47}}},\frac{1}{\frac{0}{\binom{35}{71}}},\frac{1}{\frac{0}{\binom{53}{107}}},\frac{1}{\frac{0}{\binom{80}{161}}},\frac{2}{\frac{0}{\binom{60}{121}}},\frac{2}{\frac{0}{\binom{45}{91}}},\frac{1}{\frac{0}{\binom{68}{137}}},\frac{2}{\frac{0}{\binom{51}{103}}},\frac{1}{\frac{0}{\binom{77}{155}}},\frac{1}{\frac{0}{\binom{116}{233}}},\frac{2}{\frac{0}{\binom{87}{175}}},\frac{1}{\frac{0}{\binom{131}{263}}},\frac{1}{\frac{0}{\binom{197}{395}}},\frac{1}{\frac{0}{\binom{296}{593}}},\frac{2}{\frac{0}{\binom{222}{445}}},\frac{3}{\frac{0}{\binom{83}{167}}},\frac{1}{\frac{0}{\binom{125}{251}}},\frac{1}{\frac{0}{\binom{188}{377}}},\frac{2}{\frac{0}{\binom{141}{283}}},\right.$$

$$\left.\frac{1}{\frac{0}{\binom{212}{425}}},\frac{2}{\frac{0}{\binom{159}{319}}},\frac{1}{\frac{0}{\binom{239}{479}}},\frac{1}{\frac{0}{\binom{359}{719}}},\frac{1}{\frac{0}{\binom{539}{1079}}},\frac{1}{\frac{0}{\binom{809}{1619}}},\frac{1}{\frac{0}{\binom{1214}{2429}}},\frac{3}{\frac{0}{\binom{455}{911}}},\frac{1}{\frac{0}{\binom{683}{1367}}},\frac{1}{\frac{0}{\binom{1025}{2051}}},\frac{1}{\frac{0}{\binom{1538}{3077}}},\frac{4}{\frac{0}{\binom{288}{577}}},\frac{2}{\frac{0}{\binom{216}{433}}},\frac{2}{\frac{0}{\binom{162}{325}}},\frac{4}{\frac{0}{\binom{30}{61}}},\frac{3}{\frac{0}{\binom{11}{23}}},\frac{1}{\frac{0}{\binom{17}{35}}},\frac{1}{\frac{0}{\binom{26}{53}}},\frac{5}{\frac{0}{\binom{2}{5}}},\frac{4}{\frac{0}{\binom{0}{1}}},\frac{2}{\frac{0}{\underline{\binom{0}{1}}}},\ldots\right\}$$

we can see in the iteration of 27 that, following *property 25*, after
$p_{\max}(27)=\left(\ln(27)/\ln(2)-1\right)\approx 3.75489$ iterations (3 in practice), $a_m=(C^m(n)-1)/2$,
$C^m(n)=1+2*a_m$, highlighted with the brackets, and all the following x_p are null.

13 - History limitations

If we take a number with all its iterates, up to infinity, above the limit number of iterations given by *property 25*, its x_p will be null and the trajectory will follows the arrows as in *fig. 17*.
If we start from this history of identically null x_p, and we change only one value of an iterated evenness, then we are shifted from this arrow trajectory, and at least locally $x_p\neq 0$. In general, it seems that if after that we keep the same history, then we never come back on a similar trajectory, so that the generator diverges to infinity.

A first example of such a diverging trajectory can be built on the number 1, with identical history of 2. If we change the first iterated evenness, by another number but not the same one following the first iteration periodicity (2), i.e. if we pick an odd first i but keep all the others equal to 2, $\overset{n}{H}(u_n)=\{i_1,2,\ldots i_n=2\}$.
This gives $a_1=2$ instead of 0, and $a_2=6$, $a_3=18$, $a_4=54$, as can be seen in *fig. 17*. Let us show that it is simply given by $a_p=2*3^{p-1}$ recursively. As a_p is even, we have (*prop 18*) :
$a_{p+1,1}=(1+3*a_p+3^{p+1})/2$.
If $a_p=2*3^{p-1}$, this gives $a_{p+1,1}=(1+5*3^p)/4$
Property 19 reads : $a_{p+1,2}=(a_{p+1,1}+\alpha*3^{p+1})/2+(3^{p+1}-1)/4$ with $\alpha\in\{-1,0,1\}$ depending on the parities. For $a_p=2*3^{p-1}$, this gives $a_{p+1,2}=(2*3^p+(2\alpha+2)*3^{p+1})/4$.
Let us suppose parity of (3^p-1) : $(3^p-1)\in 2^e S(1,2)$.
We know (appendix ***B***) that if $e(3^p-1)=1$ then $e(3^{p+1}-1)\geq 3$
and reciprocally if $e(3^p-1)\geq 3$ then $e(3^{p+1}-1)=1$
If we start with $3^p=1+2^e(1+2k)$

$a_{p+1,1} = (1+5*(1+2^e(1+2k)))/2 = (6+2^e*5*(1+2k))/2 = 3+2^{e-1}*5*(1+2k)$

so if $e(3^p-1)=1$, $e(3^{p+1}-1)\geq 3$ and $a_{p+1,1} = 3+5*(1+2k) = 8+10k$ so $e(a_{p+1,1})\geq 1$ and $\alpha = 0$

if $e(3^p-1)\geq 3$, $e(3^{p+1}-1)=1$ and $a_{p+1,1} = (3+5*2^{e-1})+5*2^e k \in S(1,2)//S(1+5*2^{e-2}, 5*2^{e-1})$ is odd so $e(a_{p+1,1})=0$ and $\alpha = 0$.

For $\alpha = 0$ property 19 gives $a_{p+1,2} = 2*3^p$ as guessed.

We have $x_m(h_m) = \left[-2-3*a_{m-1}(h_{m-1})+2^{i_m-1}(1+2a_m(h_m))\right]/3^m$

With $a_p = 2*3^{p-1}$ this gives

$x_m(h_m) = \left[-2-3*2*3^{m-2}+2(1+2*2*3^{m-1})\right]/3^m = 2$

So with a constant not null x the series of generators diverges very quickly.

Of course, after changing one iterated evenness, we could then follow another history, following the arrows starting from the new odd index, and we will just have a larger highest generator/initial number. But the new history is different from the first one, not only on this one, but on the next values. Another possibility is to go back to the previous series of ‘a’ later, changing just another iterated evenness to do so, and follow the same history after. This will also just have another higher generator.

This shows that even if the history $\overset{\infty}{H}$ is enough to define a number, and that from it one can deduce the series of ‘a’s $\overset{\infty}{D}$, this series together with a diagram as fig. 17 is more convenient to deduce the corresponding history but also ‘x’s series $\overset{\infty}{Q}$ and thus the series of generators. Instead of defining a number by his history $\overset{\infty}{H}$, we could thus define it by his subsetting series $\overset{\infty}{D}$. But as presented above it is better to provide the full information in parallel $\overset{\infty}{H}/\overset{\infty}{Q}/\overset{\infty}{D}$, even if of course redundant.

This easy divergence also shows that the number of possible “natural histories” is in fact very limited. Or in other words, that if a history is a sufficient condition to define a number, this number might very often be infinite. This is not surprising if one recall that for each number $n \in N$, we can read an infinite history. Reciprocally, if we write an arbitrary history, their numbers are N^N which is, following Cantor, the power of the continuum, much larger than the power of the integers. The number of possible histories is thus very limited, and this corresponds that many changes to an existing history leads to an infinite (impossible) number. Moreover, it seems that mot of the possible historied are also very similar, sharing the same end, as visible in *fig. 1*.

It would thus be interesting to see what are the actual histories, and their properties compared to any other one. This should be done in the next part.

14 - Conclusion

We first define a generalized *evenness*, the number of times an integer can be dived by 2. Applied on the Collatz iteration function, this allows to define the *history*, of an iterated number, by the series of iterated evenness.

Defining the *subsetting* operation between periodic series of integers, allows to construct a decomposition of these series along their generalized evenness (*odd-even decomposition*). This decomposition reveals always the same particular (periodic) fractal structure (*property 12*).

Describing how the periodicity of this structure is transformed by the Collatz function can be reduced to simple functions (*property 18*, *19*). The shape of these functions explains the appearent stochasticiy of the iterations. They allow to describe explicitely the iteration of this function (*property 20*), that can be summarised in simple diagram (*fig. 17*), and to obtain some results, surprising (*property 22*), or general (*property 25*). In particular we find that even if a given history is enough to built a corresponding number, only few histories can lead to finite ones.

These results might help to progress on the way of demonstrating Collatz's conjecture, or the general tools developed could be usuefull in other istuations. In the next part, we will look more in details at the possible histories.

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Appendix *A*

Let us show explicitly that a generator of double history $g_2(i, j)$ is iterated periodically into another one $g_2(j, l)$ with $l \leq 4$. We can start from the expression of $g_2(i, j)$:

$$g_2(i, j) = 1 + 2^2\left(2^{i-2}(1 + 2a(1, i)) - 1\right)/3 + 2^{i+1}\left[2\left(2^{j-2}(1 + 2a(2, j + \varphi_1(i))) - 1\right) - 3 * a(1, i)\right]/3^2$$

$$T(g_2(i, j)) = 4 + 2^2 * 3 * \left(2^{i-2}(1 + 2a(1, i)) - 1\right)/3 + 2^{i+1}\left[2\left(2^{j-2}(1 + 2a(2, j + \varphi_1(i))) - 1\right) - 3 * a(1, i)\right]/3$$

As we know it is a generator of $h = \{i,j\}$ so the first iteration will give to a division by 2^i, and indeed we find :

$$C(g_2(i, j)) = T(g_2(i, j))/2^i = 1 + 2^2\left(2^{j-2}(1 + 2a(2, j + \varphi_1(i))) - 1\right)/3$$

This is a similar expression than the beginning of $g_2(i, j)$, except *i* has been transformed into *j* (as guessed) and $a(1, i)$ into $a(2, j + \varphi_1(i))$. Let us show explicitly that it is exactly another generator $g_2(j, l)$:

$$C(g_2(i, j)) = 1 + 2^2\left(2^{j-2}(1 + 2a(2, j + \varphi_1(i))) - 1\right)/3$$

$$g_2(j, l) = 1 + 2^2\left(2^{j-2}(1 + 2a(1, j)) - 1\right)/3 + 2^{j+1}\left[2\left(2^{l-2}(1 + 2a(2, l + \varphi_1(j))) - 1\right) - 3 * a(1, j)\right]/3^2$$

Equalling the two give the condition:

$$2 + 3 * a(2, j + \varphi_1(i)) = 2^{l-1}(1 + 2 * a(2, l + \varphi_1(j)))$$

Let us recall that $a(2, j) = \left\{\underline{8, 6, 5, 0, 2, 3,} ...\right\}$, $\varphi(i) = \left\{\underline{0, 2,} ...\right\}$.

For *i* odd $\varphi_1(i) = 0$, and for *j* odd the condition reads $3 * a(2, j) = -2 + 2^{l-1} + 2^l a(2, l)$,

we can check that it is satisfied for $l(jodd) = \left\{\underline{2, 1, 4,} ...\right\}$.

For *j* even we have $3 * a(2, j) = -2 + 2^{l-1} + 2^l a(2, l + 2)$,

and the solutions are $l(jeven) = \left\{\underline{3, 2, 1,} ...\right\}$

For *i* even $\varphi_1(i) = 2$ so we have the same solutions except shifted by 2.

So the equality is actually true with, for *i* odd $l(j) = \left\{\underline{2, 3, 1, 2, 4, 1,} ...\right\}$,

and for *i* even the same series shifted by 2, $l(j) = \left\{\underline{1, 2, 4, 1, 2, 3,} ...\right\}$.

This corresponds to the period 6 pattern indicated in bold in *fig. 10*.

Appendix ***B***

One can simply demonstrate the period 2 for the evenness 1 (or higher than 1) of 3^n-1 by iteration. We can first notice that
$3^{n+1}-1=3*(3^n-1)+2$
so we can define the new function:
$R(a)=3*a+2$
If we assume that $e(3^n-1)=1$, it means we start from $(3^n-1)\in 2*S(1,2)$. We can then apply R to it:
$R(2*S(1,2))=R(S(2,4))=S(8,12)=4*S(2,3)$
with $S(2,3)$ decomposing on all the evenness (*property 9*), so $e(3^{n+1}-1)\geq 2$, and
$R^2(2*S(1,2))=R(S(8,12))=S(26,36)=2*S(13,18)=2*S(1,2)//S(6,9)$
is again only of evenness 1.

Starting with evenness 3:
$e(3^n-1)=3$, so $(3^n-1)\in 2^3*S(1,2)$ and we have
$R(2^3*S(1,2))=R(S(8,16))=S(26,48)=2*S(13,24)=2*S(1,2)//S(6,12)$
is indeed of evenness 1, and the next one
$R^2(2^3*S(1,2))=R(S(26,48))=S(80,144)=2^4*S(5,9)$
which contains all the evenness larger or equal to 4,
The next two iterations gives
$R^3(2^3*S(1,2))=R(S(80,144))=S(242,432)=2*S(121,216)=2*S(1,2)//S(60,108)$
is again only of evenness 1, while
$R^4(2^3*S(1,2))=R(S(242,432))=S(728,1296)=2^3*S(91,162)=2^3*S(1,2)//S(45,81)$
is again only of evenness 3.

For evenness 4:
$e(3^n-1)=4$, so $(3^n-1)\in 2^4*S(1,2)$ and we have
$R(2^4*S(1,2))=S(50,96)=2*S(1,2)//S(12,24)$
$R^2(2^4*S(1,2))=S(152,288)=2^3*S(1,2)//S(9,18)$
$R^3(2^4*S(1,2))=S(458,864)=2*S(1,2)//S(114,216)$
$R^4(2^4*S(1,2))=S(1376,2592)=2^5*S(43,81)$ of evenness ≥ 5
$R^5(2^4*S(1,2))=S(4130,7776)=2*S(1,2)//S(1032,1944)$
$R^6(2^4*S(1,2))=S(12392,23328)=2^3*S(1,2)//S(774,1458)$
$R^7(2^4*S(1,2))=S(37178,69984)=2*S(1,2)//S(9294,17496)$
$R^8(2^4*S(1,2))=S(111536,209952)=2^4*S(1,2)//S(3485,6561)$